EXISTENCE AND MULTIPLICITY RESULTS FOR p-q-LAPLACIAN BOUNDARY VALUE PROBLEMS

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Dedicated to the living memory of Alan C. Lazer

Abstract. We study positive solutions to the boundary value problem

\[-\Delta_p u - \Delta_q u = \lambda f(u) \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega,\]

where \(q \in (1, p)\) and \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(N > 1\) with smooth boundary, \(\lambda\) is a positive parameter, \(\Delta_p\) and \(\Delta_q\) are the \(p\)-Laplacian and \(q\)-Laplacian operators, respectively, and \(f : [0, \infty) \to (0, \infty)\) is \(C^1\), nondecreasing, and \(p\)-sublinear at infinity i.e. \(\lim_{t \to \infty} \frac{f(t)}{t^{p-1}} = 0\). We discuss existence and multiplicity results for classes of such \(f\). Further, when \(N = 1\), we discuss an example which exhibits \(S\)-shaped bifurcation curves.

1. Introduction

In [12], the authors studied the the \(p\)-Laplacian boundary value problem

\[-\Delta_p u = \lambda f(u) \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega\]

where \(p > 1\), \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(N > 1\) with smooth boundary, \(\lambda\) is a positive parameter, \(\Delta_p u = \text{div}(\nabla |\nabla u|^{p-2}\nabla u)\), \(p > 1\) and \(f : [0, \infty) \to (0, \infty)\) satisfies

(H1) \(f\) is a \(C^1\) non-decreasing, \(p\)-sublinear function at infinity, i.e.

\(\lim_{t \to \infty} \frac{f(t)}{t^{p-1}} = 0\).

In particular, they established the existence of a positive solution for each \(\lambda\), and when there exists \(0 < a < b\) such that

\(Q(a, b) := \frac{a^{p-1}/f(a)}{b^{p-1}/f(b)} \gg 1,\)

they obtained multiple positive solutions for certain range of \(\lambda\). When \(p = 2\), these results were discussed in [4]. Their study was motivated by the applications in chemical reaction theory (see [2]) and in combustion theory (see [11, 14]).
In this article, we extend this study to the $p$-$q$-Laplacian boundary value problem
\begin{equation}
-\Delta_p u - \Delta_q u = \lambda f(u) \quad \text{in } \Omega,
\end{equation}
\begin{equation}
u = 0 \quad \text{on } \partial \Omega,
\end{equation}
for $q \in (1, p)$. Namely, we establish

**Theorem 1.1.** Assume (H1). Then the following results hold:

1. Equation (1.2) has a positive solution for each $\lambda > 0$.
2. There exists positive constants $C_0 = C_0(\Omega)$, $C_1 = C_1(p, \Omega, N)$, and $C_2 = C_2(\Omega)$ such that if $b > C_0$ and
   \[ Q(a, b) = \frac{a/\lambda^p}{f(a)} > \frac{C_1}{C_2} \]
   for some points $a$ and $b$, $a < b$, then (1.2) has at least two positive solutions for $\lambda \in \left(\frac{b^{p-1}}{f(b)}, \frac{a^{p-1}}{f(a)}\right)$.

As in [4] and [12] we use the method of sub-super solutions to establish our results. In [4], the availability of Green’s function played an important role in the construction of a positive strict sub-solution that was used to establish a multiplicity result. In [12], the authors had to develop another idea to construct this special sub-solution because of the lack of a Green’s function for the $p$-Laplacian operator for $p \neq 2$. Here we adapt and extend the ideas used in [12] to establish our results. However, unlike in [12], our multiplicity result is restricted to only two solutions. In [12], the authors used results in [11, 15] to guarantee three solutions. If the results in [11, 15] can be extended to the $p$-$q$ Laplacian case, our construction of sub-super solutions will also yield at least three positive solutions for the range of $\lambda$ in Theorem 1.1 part (2).

A time-dependent version of an operator such as in (1.2) often occurs in the mathematical modeling of chemical reactions and plasma physics. In recent years, a lot of attention has been given to study the boundary value problem involving $p$-$q$ Laplacian, see for instance [3, 5, 10, 13] and the references therein.

The rest of this article is organized as follows. In Section 2 we recall some important results that are required for the development of this article. In Section 3 we prove of Theorem 1.1. In Section 4 we provide an application of our results. Finally in Section 5 we obtain exact bifurcation diagrams for the case when $\Omega = (0, 1), p = 4$ and $q = 2$, namely to the two-point boundary value problem
\begin{equation}
-[(u')^3]' - \mu [(u')'] = \lambda f(u); \quad (0, 1)
\end{equation}
\begin{equation}
u(0) = 0 = u(1)
\end{equation}
where $f(s) = \exp (\frac{\gamma s}{\gamma + s}), \gamma > 0$, and $\mu$ is a non-negative parameter.

2. Preliminaries

In this section, we recall some results concerning a sub-super solution method for $p$-$q$-Laplacian boundary value problem. First, by a weak solution of (1.2) we mean a function $u \in W_0^{1,p}(\Omega)$ which satisfies
\begin{equation}
\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \phi + \int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla \phi = \lambda \int_\Omega f(u) \phi, \quad \forall \phi \in C_0^\infty(\Omega).
\end{equation}
However, in this article, we in fact study $C^1(\Omega)$ solution. Next, by a sub-solution (super solution) of (1.2) we mean a function $v \in W^{1,p}(\Omega) \cap C(\Omega)$ such that $v \leq (\geq)0$ on $\partial \Omega$ and satisfies

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \phi + \int_{\Omega} |\nabla v|^{q-2} \nabla v \cdot \nabla \phi \leq (\geq) \lambda \int_{\Omega} f(v) \phi,$$

for all $\phi \in C_0^\infty(\Omega)$ and $\phi \geq 0$ in $\Omega$. Then the following sub-super solution result holds.

**Lemma 2.1.** Let $\psi, z$ be sub and super solutions of (1.2) respectively such that $\psi \leq z$ in $\Omega$. Then (1.2) has a solution $u \in C^1(\Omega)$ such that $\psi \leq u \leq z$.

For a proof of the above lemma see [7, Corollary 1].

### 3. Proof of Theorem 1.1

In this section we use sub-super solution method to prove Theorem 1.1. At first we prove the results when $\Omega = B_R$, a ball of radius $R$ and centered at origin in $\mathbb{R}^N$. We adopt and extend the ideas presented in [12] to construct a crucial sub-solution on $B_R$.

**Construction of two sub-solutions on $B_R$.** Clearly $\phi_1 = 0$ is a sub-solution to the problem (1.2). Now we construct another sub-solution. For that we consider the function

$$v(r) = \begin{cases} 1, & r \leq \epsilon \\ 1 - \left(1 - \frac{R-r}{R-\epsilon}\right)^\alpha, & \epsilon \leq r \leq R, \end{cases}$$

where $\epsilon \in (0, R)$, $\alpha > 1$ and $\beta > 1$. Let us denote $\mu_1(r) = \frac{R-r}{R-\epsilon}$ and $\mu_2(r) = 1 - (\mu_1(r))^{\beta}$. Taking $\tilde{\psi}(r) = b\psi(r)$ we note that $|\tilde{\psi}'(r)| \leq b\alpha\beta/(R-\epsilon)$. Now let $\psi$ be a radially symmetric solution of

$$-\Delta_p \psi - \Delta_q \psi = \lambda f(\psi(|x|)) \quad \text{in } B_R,$$

$$\psi = 0 \quad \text{on } \partial B_R. \quad (3.1)$$

Then $\psi$ satisfies

$$-(r^{N-1}G_p(\psi'(r)))' - (r^{N-1}G_q\psi'(r))' = \lambda r^{N-1}f(\tilde{\psi}(r))$$

$$\psi'(0) = \psi(R) = 0, \quad (3.2)$$

where $G_s(t) = |t|^{s-2}t$, $s = p, q$. Integrating the above equation over $0 < r < R$ we obtain

$$-G_p(\psi'(r)) - G_q\psi'(r) = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1}f(\tilde{\psi}(s)) \, ds. \quad (3.3)$$

Notice that $\hat{G}(t) := G_p(t) + G_q(t)$ is a continuous, monotone function. Hence, $\hat{G}^{-1}$ exists and is also continuous. Therefore, (3.3) yields

$$\psi'(r) = \hat{G}^{-1}\left(\frac{\lambda}{r^{N-1}} \int_0^r s^{N-1}f(\tilde{\psi}(s)) \, ds\right). \quad (3.4)$$

Next, we claim that $\tilde{\psi}(r) \leq \psi(r)$, for $0 \leq r \leq R$. If this claim is true, then $\psi$ is a sub-solution as $f$ is nondecreasing. Now, since $\psi(R) = \psi(R) = 0$, it is sufficient to show that $\psi'(r) \leq \tilde{\psi}'(r)$ for all $0 \leq r \leq R$. Observe that $\psi'(r) = 0$ for $0 \leq r \leq R$ and $\tilde{\psi}'(r) = 0$ for $0 \leq r \leq \epsilon$. Where as for $r \geq \epsilon$ we have

$$\int_0^r s^{N-1}f(\tilde{\psi}(s)) \, ds \geq \int_0^\epsilon s^{N-1}f(\tilde{\psi}(s)) \, ds \geq f(b) \frac{\epsilon^N}{N}.$$
It follows from (3.4) that

$$-\psi'(r) \geq \tilde{G}^{-1}\left(f(b)\frac{\lambda e^N}{N R^{N-1}}\right).$$

Recall that $|\tilde{v}'(r)| \leq \frac{b \alpha \beta}{R - \epsilon}$. Thus, $\psi'(r) \leq v'(r)$ if $\tilde{G}^{-1}(f(b)\frac{\lambda e^N}{N R^{N-1}}) \geq \frac{b \alpha \beta}{R - \epsilon}$ i.e. if

$$f(b)\frac{\lambda e^N}{N R^{N-1}} \geq \tilde{G}(\frac{b \alpha \beta}{R - \epsilon}).$$

(3.5)

Note that (3.3) will be satisfied if

$$f(b)\frac{\lambda e^N}{N R^{N-1}} \geq \max\{2, 2G_p(\frac{b \alpha \beta}{R - \epsilon})\},$$

i.e. if

$$\lambda \geq \frac{2 N R^{N-1}}{f(b)\epsilon^N} \max\{1, (\frac{b \alpha \beta}{R - \epsilon})^{p-1}\}.$$  

(3.6)

Let $b > R$. Then we can choose $\alpha \approx 1$, $\beta \approx 1$ so that $(\frac{b \alpha \beta}{R - \epsilon}) > 1$ and (3.6) will be satisfied if

$$\lambda \geq \frac{b^{p-1}}{f(b)} C_1 (\alpha \beta)^{p-1},$$

(3.7)

where $C_1 = \inf_{\mathbb{R}^+} \frac{2 N R^{N-1}}{\lambda^{1/p}(R - \epsilon)^{p-1}}$. In fact, this infimum is achieved at $\epsilon = \epsilon_0 = \frac{N R}{N + p - 1}$, which will be our choice of $\epsilon$. Assume that

$$\lambda > \frac{b^{p-1}}{f(b)} C_1.$$  

(3.8)

Then clearly we can fix $\alpha(> 1) \approx 1$, $\beta(> 1) \approx 1$ so that (3.7) holds. Hence, for these choice of $\alpha, \beta, \epsilon, \psi$ will be a sub-solution, when (3.8) is satisfied. Further, since $\psi \geq \tilde{v}, \|\psi\|_{\infty} \geq 2$.

Construction of super solutions on $B_R$. Let $\sigma(r) = (1 - (\frac{r}{p})^{p'})/p'$ on $B_R$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Notice that $0 \leq \sigma \leq 1$. Also that for $0 \leq r \leq R$,

$$\sigma'(r) = -\frac{r^{p'-1}}{R^{p'}};$$

$$-(r^{N-1}G_p(\sigma'(r)))' = -(r^{N-1}|\sigma'(r)|^{p'-2}\sigma'(r))' = \left(\frac{r^{N-1}p(p'-1)(s-1)}{R^{p'(s-1)}}\right)' \geq 0,$$

(3.9)

In particular, $-(r^{N-1}G_p(\sigma'(r)))' = \frac{N+1}{R^{p'}}$. Now let $\xi_a = \frac{N+1}{R^{p'}} a \sigma$, where $a$ as in the assumption of Theorem 1.1. Then, since $f$ is nondecreasing,

$$-(r^{N-1}G_p(\xi_a'(r)))' - (r^{N-1}G_p(\xi_a'(r)))' \geq \frac{\sigma^{N-1}a^{p-1}}{R^{p'}} \geq \lambda r^{N-1}f(\xi_a(r))$$

if $\lambda \leq \frac{a^{p-1}}{f(a)R^p}$. Thus, $\bar{\xi}_a := \xi_a(|x|)$ satisfies

$$-\Delta_p \bar{\xi}_a - \Delta_q \bar{\xi}_a \geq \frac{a^{p-1}}{R^{p'}} \geq \lambda f(\bar{\xi}_a) \quad \text{if} \quad \lambda \leq \frac{a^{p-1}}{f(a)R^p}.$$  

(3.10)

Hence, $\bar{\xi}_a$ is a super solution when $\lambda \leq \frac{a^{p-1}}{f(a)R^p}$. Next for a given $\lambda > 0$, let $\bar{\xi}_a = N^{\frac{1}{p'}} M(\lambda) \sigma(|x|)$, where $M(\lambda) >> 1$ so that $\frac{|m(\lambda)|^{p-1}}{f(M(\lambda))} \geq \lambda R^p$. Then, again using $f$ is nondecreasing we have

$$-\Delta_p \bar{\xi}_a - \Delta_q \bar{\xi}_a \geq \frac{M(\lambda)^{p-1}}{R^{p'}} \geq \lambda f(\bar{\xi}_a),$$  

(3.11)
hence \( \bar{\xi}_\lambda \) is a super solution on \( B_R \).

**Comparison of Sub-Sup solutions on \( B_R \).** For any \( \lambda > 0 \), \( \psi_1 \equiv 0 \) is a strict sub-solution (as \( f(0) > 0 \)) and \( z_2 = \bar{\xi}_\lambda = N \frac{1}{\lambda^{p-1}} M(\lambda) \sigma(|x|) \) with \( M(\lambda) >> 1 \) is a super solution. Hence, by Lemma 2.1, (1.2) has a positive solution for each \( \lambda > 0 \). Next let \( \lambda \in (\frac{\phi^{p-1}}{f(0)} C_1, \frac{\phi^{p-1}}{f(0)} C_2] \), where \( C_2 = \frac{1}{R^p} \) and \( 0 < a < b \) such that \( b > R = C_0(\Omega) \) and \( Q(a, b) \geq \frac{C_1}{C_2} \). For such \( \lambda \), \( \psi_1 \equiv 0 \) is a strict sub-solution, \( \psi_2 = \psi (\psi \text{ as in (3.1)} \) is a sub-solution, \( z_1 = \tilde{\xi}_\alpha = N \frac{1}{\lambda^{p-1}} a \sigma(|x|) \) is a super solution (see (3.10)) and \( z_2 = \bar{\xi}_\lambda = N \frac{1}{\lambda^{p-1}} M(\lambda) \sigma(|x|) \) is a super solution. Hence, by Lemma 2.1, (1.2) has two solutions \( u_1, u_2 \) for such \( \lambda > 0 \), where \( u_1 \in [\psi_1, z_1] \) and \( u_2 \in [\psi_2, z_2] \). Note that, \( u_1 \) and \( u_2 \) are distinct since \( \|z_1\|_\infty \leq a \), \( \|\psi_2\|_\infty \geq b \) and \( a < b \).

Now we proceed to prove our result for any open, bounded subset \( \Omega \) of \( \mathbb{R}^N \).

**Proof of Theorem 1.1.** Note that \( \psi_1 = 0 \) still remains a sub-solution on \( \Omega \) to (1.2) on \( \Omega \). Now let \( B_R \) be the largest inscribed ball inside \( \Omega \) and we define

\[
\tilde{\psi}_2(x) = \begin{cases} 
\psi(x), & \text{if } x \in B_R \\
0, & \text{if } x \in \overline{\Omega} \setminus B_R
\end{cases}
\]

where \( \psi \) is as in (3.1). Clearly, \( \tilde{\psi}_2 \in W^{1,p}_0(\Omega) \) and when \( \lambda > C_1 b^{p-1} / f(b) \) we have

\[-\Delta_p \tilde{\psi}_2(x) = -\Delta_p \psi_2(x) \leq \lambda f(\psi_2(x)) = \lambda f(\tilde{\psi}_2(x)) \quad \text{on } B_R.\]

Also \( -\Delta_p \tilde{\psi}_2(x) = 0 < \lambda f(\lambda f(\psi_2(x))) \) in \( \Omega \setminus B_R \). Hence, \( \tilde{\psi}_2 \) is a strict sub-solution when \( \lambda > C_1 b^{p-1} / f(b) \). Also \( \|\tilde{\psi}_2\|_\infty \geq b \). Next, we consider a ball \( B_{\Pi} \) containing \( \Omega \). By taking \( z_1 = \tilde{\xi}_\alpha = N \frac{1}{R^{p-1}} a \sigma(|x|) \) and \( z_2 = \bar{\xi}_\lambda = N \frac{1}{\lambda^{p-1}} M(\lambda) \sigma(|x|) \) as earlier (but now in ball \( B_{\Pi} \)) and taking their restrictions on \( \Omega \), it is easy to see that \( z_1 \) is a strict super solution (1.2) on \( \Omega \) if \( \lambda \leq \frac{\alpha^{p-1}}{b^{p-1}} \), while \( z_2 = \bar{\xi}_\lambda \) with \( M(\lambda) >> 1 \) is a super solution to (1.2) on \( \Omega \) for any \( \lambda > 0 \). Noticing again \( \|z_1\|_\infty \leq a \) and using Lemma 2.1 the proof of Theorem 1.1 follows in the general region \( \Omega \). \( \square \)

**Remark 3.1.** Under assumption (H1), if \( f(s) / s^{q-1} \) is strictly decreasing for \( s > 0 \), then (1.2) has a unique positive solution \( [8, \text{Theorem 2.2}] \).

4. **APPLICATION IN COMBUSTION THEORY**

For \( 1 < q < p \), we consider

\[-\Delta_p u - \Delta_q u = \lambda \exp \left( \frac{\gamma u}{\gamma + u} \right) \quad \text{in } \Omega, \]

\[u = 0 \quad \text{on } \partial \Omega.\]

The reaction term \( f(s) = \exp \left( \frac{\gamma u}{\gamma + u} \right), \gamma > 0 \) occurs in the theory of combustion and it has been discussed in [1] (Laplacian case), and in [12] (\( p \)-Laplacian case). In [4], the authors obtained that \( \gamma > 4 \) is a necessary condition for multiple positive solutions for the Laplacian case; while in [12] the authors obtained the same for \( \gamma > 4(p-1) \) in the \( p \)-Laplacian case. Here we present analogous result for \( p-q \) Laplacian. Towards this we first notice that \( \hat{f}(u) := f(u) / u^{q-1} \) is decreasing if \( \gamma \leq 4(q-1) \). Thus, Remark 3.1 ensures that \( \gamma > 4(q-1) \) is a necessary condition for (4.1) to have multiple solutions. Further, taking \( a = 1 \) and \( b = \gamma \), we have

\[Q(a, b) := \frac{\alpha^{p-1}}{b^{p-1}} / f(b) = \gamma (1-p) \exp \left( \frac{\gamma}{2} - \frac{\gamma}{\gamma + 1} \right).\]
Thus, for any $C_0, C_1$ and $C_2$, we can choose $\gamma$ large enough such that $b > C_0 Q(1, \gamma) > \frac{C_2}{C_1}$ and hence, by Theorem 1, admits at least two solutions at least for certain range of $\lambda$.

5. Bifurcation Diagram for Positive Solutions to (5.1)

Here we study the two-point boundary value problem

$$
-[(u')^3]' - \mu [(u')'] = \lambda f(u); \quad (0, 1)
$$

$$
u(0) = 0 = u(1)
$$

(5.1)

where $f(s) = \exp \left( \frac{2s}{\gamma + s} \right)$; $\gamma > 0$, and $\mu$ is a non-negative parameter. We will provide the exact bifurcation diagram via a quadrature method and Mathematica computations. We will also study how this bifurcation curve evolves when $\gamma$ and $\mu$ vary.

Here we use the quadrature method described in [6] which was obtained by extending the method initially introduced in [9]. First we note that since (5.1) is autonomous, any positive solution $u$ must be symmetric about $x = 1/2$, increasing on $(0, 1/2)$, and decreasing on $(1/2, 1)$. Assume $u$ is a positive solution of (5.1) and let $u(1/2) = \rho$.

**Figure 1.** Shape of a positive solution to (5.1)

Multiplying (5.1) by $u'$ and integrating we obtain

$$
-\frac{3}{4} ((u')^4)' - \mu [(u')'] = \lambda (F(u))' \quad \text{in} \quad (0, 1),
$$

where $F(s) = \int_0^s f(z)dz$. Further integrating we obtain

$$
3[u'(x)]^4 + 2\mu[u'(x)]^2 = 4\lambda[F(\rho) - F(u(x))] \quad \text{in} \quad [0, \frac{1}{2}],
$$

and hence

$$
u'(x) = \frac{\sqrt{\mu^2 + 12\lambda(F(\rho) - F(u(x)))}}{\sqrt{3}} - \mu \quad \text{in} \quad [0, \frac{1}{2}],
$$

(5.2)

Integrating (5.2), we obtain

$$
\int_0^{u(x)} \frac{ds}{\sqrt{\mu^2 + 12\lambda(F(\rho) - F(s))}} = \frac{x}{\sqrt{3}} \quad \text{in} \quad [0, \frac{1}{2}],
$$

(5.3)
and setting $x \to \left(\frac{1}{2}\right)^{-}$, we obtain

$$G(\lambda, \rho) = \int_{0}^{\rho} \frac{ds}{\sqrt{[\mu^2 + 12\lambda(F(\rho) - F(s))]^2 - \mu}} = \frac{1}{2\sqrt{3}}.$$  \hspace{1cm} (5.4)

Inversely, if $\lambda, \rho$ are such that (5.4) is satisfied, $u(x)$ is defined via (5.3) for $x \in [0, \frac{1}{2}), u(1/2) = \rho$, and $u(x) = u(1 - x)$ for $x \in (1/2, 1]$, it follows that $u$ will be a positive solution of (5.1). Hence (5.4) determines the bifurcation diagram of positive solutions for (5.1). Now, for $f(s) = \exp\left(\frac{\gamma s}{\gamma + s}\right)$, we use Mathematica computations to obtain the bifurcation diagram using (5.4).

**Observations.** For a given $\mu \geq 0$, there exists $\gamma_0(\mu)$ such that for $\gamma < \gamma_0(\mu)$, we obtain a unique solution of (5.1) for all $\lambda > 0$, and for $\gamma > \gamma_0(\mu)$ the bifurcation curve is $S$-shaped with multiplicity in the region $(\lambda_1, \lambda_2)$. Further, $\gamma_0(\mu)$ decreases in $\mu$ and $\lambda_1$ decreases in $\gamma$ (see Figure 2). Furthermore, strength of the multiplicity range (i.e. the length $(\lambda_2 - \lambda_1)$) increases in $\gamma$ (See Figure 3).

![Figure 2. Bifurcation diagrams of (5.1) for different values of $\gamma$ for given $\mu$.](image)

![Figure 3. Heat map showing the strength of multiplicity range w.r.t $\gamma$ and $\mu$.](image)
References


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