COMPLETE CLASSIFICATION OF BIFURCATION CURVES FOR
A MULTIPARAMETER DIFFUSIVE LOGISTIC PROBLEM WITH
GENERALIZED HOLLING TYPE-IV FUNCTIONAL RESPONSE

JYUN-YUAN CIOU, TZUNG-SHIN YEH

Abstract. We study exact multiplicity and bifurcation curves of positive solutions for the diffusive logistic problem with generalized Holling type-IV functional response

\[ u''(x) + \lambda [ru(1 - \frac{u}{q}) - \frac{u}{1 + mu + u^2}] = 0, \quad -1 < x < 1, \]
\[ u(-1) = u(1) = 0, \]

where the quantity in brackets is the growth rate function and \( \lambda > 0 \) is a bifurcation parameter. On the \((\lambda, \|u\|_\infty)\)-plane, we give a complete classification of two qualitatively different bifurcation curves: a C-shaped curve and a monotone increasing curve.

1. Introduction

We study the exact multiplicity and bifurcation curves of positive solutions for the diffusive logistic problem with generalized Holling type-IV functional response,

\[ u''(x) + \lambda [ru(1 - \frac{u}{q}) - \frac{u}{1 + mu + u^2}] = 0, \quad -1 < x < 1, \]
\[ u(-1) = u(1) = 0, \]

where the growth rate function is \( f(u) = ug(u) \) with

\[ g(u) = r\left(1 - \frac{u}{q}\right) - \frac{1}{1 + mu + u^2}, \]

\( m \geq 1, \; q, \; r \) are two positive dimensionless parameters, and \( \lambda > 0 \) is a bifurcation parameter.

In population dynamics, a functional response of the predator to the prey density depends on the change in the density of prey susceptible to each predator per unit time. The simplest response function is

\[ p_1(u) = \begin{cases} au, & 0 \leq u < k/a, \\ k, & u \geq k/a, \end{cases} \]

2010 Mathematics Subject Classification. 34B15, 34B18.
Key words and phrases. Bifurcation curve; exact multiplicity; diffusive logistic problem; Holling type-IV functional response; time map.
©2021 Texas State University.
where \( k, a > 0 \), which is called Holling type-I function in [6]. Michaelis and Menten proposed the response function
\[
p_2(u) = \frac{cu}{a + u},
\]
in the studying enzymatic reactions, where \( a, c > 0 \), which is called Holling type-II function in [6]. Another class of response function is
\[
p_3(u) = \frac{cu^2}{a + u^2},
\]
where \( a, c > 0 \), it is known as a Holling type-III function. Wang and Yeh [24] studied a multiparameter diffusive logistic problem with Holling type-III functional response. Note that \( p_1(u), p_2(u), p_3(u) \) are monotonic on \((0, \infty)\). Sokol and Howell [19] proposed a non-monotonic response function
\[
p_4(u) = \frac{cu}{a + u^2},
\]
which is called the simplified Holling type-IV function. This case has been extensively studied by many authors, see Baek [1], Li and Xiao [9], Lian and Xu [10], Qolizadeh Amirabad et al. [16], Ruan and Xiao [17], and Yeh [25]. Another class of non-monotonic response functions is the generalized Holling type-IV function
\[
\tilde{p}_4(u) = \frac{cu}{a + bu + u^2},
\]
where \( a, c > 0 \) and \( b \geq 0 \). The response function \( \tilde{p}_4(u) \) satisfies \( \tilde{p}_4'(u) > 0 \) on \((0, \sqrt{a})\) and \( \tilde{p}_4'(u) < 0 \) on \((\sqrt{a}, \infty)\). Collings [4] used the response function \( \tilde{p}_4(u) \) in a mite predator-prey interaction model for \( b \geq \sqrt{a} \). The generalized Holling type-IV function has been studied by Huang and Xiao [7], Liu and Huang [11], and Upadhyay et al. [21].

The idea of using diffusion to study population dynamics was introduced by Skellam [18] in the early 1950s. Since then, reaction-diffusion equations have been widely used for the formation of spatial population patterns and the description of the effects of organisms’ spatial dispersal in population dynamics; see Britton [2], Cantrell and Cosner [3], Fife [5], Murray [14], and Okubo [15]. Sounvoravong et al. [20] studied a reaction-diffusion system for a SIRS epidemic model. Problem (1.1) is motivated by the reaction-diffusion population model
\[
\frac{\partial N}{\partial T} = D \frac{\partial^2 N}{\partial X^2} + r_N N \left( 1 - \frac{N}{K_N} \right) - \frac{cN}{a + bN + N^2}, \quad \frac{L}{2} \sqrt{\frac{D}{r_N}} < X < \frac{L}{2} \sqrt{\frac{D}{r_N}}, \quad (1.3)
\]
where \( T > 0, D > 0 \) is the diffusion constant, \( N \) is the prey population density, \( r_N \) is the intrinsic growth rate of the prey population, \( K_N \) is the carrying capacity, and \( a, c > 0, b \geq 0 \). The second term on the right-hand side of (1.3) is a logistic term. The third term on the right-hand side of (1.3) gives the rate of consumption of prey by predators, which is called the predation term; see Ludwig et al. [12, 13].

We consider the problem (1.3) with
\[
w = \frac{N}{\sqrt{a}}, \quad \ddot{t} = r_N T, \quad \ddot{x} = \sqrt{\frac{r_N}{D}} X, \quad r = \frac{r_N a}{c}, \quad q = \frac{K_N}{\sqrt{a}}, \quad m = \frac{b}{\sqrt{a}}.
\]
Then problem (1.3) takes the form
\[
\frac{\partial w}{\partial \ddot{t}} = \frac{\partial^2 w}{\partial \ddot{x}^2} + w \left( 1 - \frac{w}{q} \right) - \frac{1}{r} \frac{w}{1 + mw + w^2}, \quad -\frac{L}{2} < \ddot{x} < \frac{L}{2}, \quad \ddot{t} > 0. \quad (1.4)
\]
Assume that a habitat $-L/2 \leq \tilde{x} \leq L/2$ is surrounded by a totally hostile, outer environment. That is, equation (1.4) holds in the strip $|\tilde{x}| < L/2$ and
\[
w(-L/2, \tilde{t}) = w(L/2, \tilde{t}) = 0, \quad \tilde{t} > 0.
\] (1.5)

Let $v(x, t) = w(\tilde{x}, \tilde{t})$ with $x = 2\tilde{x}/L$, $t = 4\tilde{t}/L^2$, and let
\[
\lambda = \frac{L^2}{4r}.
\]

Then problem (1.4), (1.5) takes the form
\[
\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \lambda \left[ rv(1 - \frac{v}{q}) - \frac{v}{1 + mv + v^2} \right], \quad -1 < x < 1, \quad t > 0,
\]
\[
v(-1, t) = v(1, t) = 0, \quad t > 0.
\] (1.6)

Let $u(x)$ denote a positive steady-state population density of (1.6). Then $u(x)$ satisfies (1.1).

For $m = 0$, problem (1.1) takes the form
\[
u''(x) + \lambda \left[ ru(1 - \frac{u}{q}) - \frac{u}{1 + u^2} \right] = 0, \quad -1 < x < 1,
\]
\[
u(-1) = \nu(1) = 0.
\] (1.7)
Applying the quadrature method (time-map method), Yeh [25] proved that either \( r \leq \eta_1 q \) and \((q, r)\) lies above the curve
\[
\Gamma = \left\{ (q, r) : q(a) = \frac{1 + 3a^2}{2a}, r(a) = \frac{1 + 3a^2}{(1 + a^2)^2}, 0 < a < 1/\sqrt{3} \right\}
\]
or \( r \leq \eta_2 q \) for some constants \( \eta_1 \approx 0.618 \) and \( \eta_2 \approx 0.601 \). Then on the \((\lambda, \|u\|_\infty)\)-plane, he gave a classification of four qualitatively different bifurcation curves: an S-shaped curve, a broken S-shaped curve, a C-shaped curve and a monotone increasing curve, see [25, Theorems 2.1–2.4] and Figure 1.

![Figure 2. Classified graphs of growth rate per capita \( g(u) = r(1 - \frac{u^q}{q}) - \frac{1}{1 + mu + u^2} \) with \( m \geq 1 \) on \((0, \infty)\), drawn on the first quadrant of \((q, r)\)-parameter plane.](image)

In this article we study exact multiplicity of positive solutions and shapes of bifurcation curves of (1.1) for parameters \( m \geq 1 \) and \( q, r > 0 \). We first find the number of positive zeros of growth rate per capita
\[
g(u) = r(1 - \frac{u^q}{q}) - \frac{1}{1 + mu + u^2}.
\]
Then we give a classification of \( g(u) \) on the first quadrant of \((q, r)\)-parameter plane according to the monotonicity of \( g(u) \). We divide the first quadrant of \((q, r)\)-parameter plane into the disjoint union of three curves \( \Gamma_1, \Gamma_2, \Gamma_3 \) and five regions...
$R_1, R_2, R_3, R_4, R_5$ defined as follows:

\[ \Gamma_1 = \{(q, r) : q(a) = \frac{1 + 2ma + 3a^2}{m + 2a}, r(a) = \frac{1 + 2ma + 3a^2}{(1 + ma + a^2)^2}, 0 < a < \infty \}, \]

\[ \Gamma_2 = \{(q, r) : r = mq > 1 \}, \]

\[ \Gamma_3 = \{(q, r) : 0 < r = mq \leq 1 \}, \]

\[ R_1 = \{(q, r) : 0 < r < mq \text{ and } r \geq 1 \}, \]

\[ R_2 = \{(q, r) : 0 < r < mq \text{ and } (q, r) \text{ lies between the curve } \Gamma_1 \text{ and the line } r = 1 \}, \]

\[ R_3 = \{(q, r) : 0 < r < mq \text{ and } (q, r) \text{ lies below the curves } \Gamma_1 \text{ and } \Gamma_3 \}, \]

\[ R_4 = \{(q, r) : r > mq > 0 \text{ and } r > 1 \}, \]

\[ R_5 = \{(q, r) : r > mq > 0 \text{ and } r \leq 1 \}. \]

It is well known that the curve $\Gamma_1$ is continuous and strictly decreasing on the $(q, r)$-plane. Note that, for $(q(a), r(a)) \in \Gamma_1$, $\lim_{a \to 0^+} q(a) = \frac{1}{m}$ and $\lim_{a \to 0^+} r(a) = 0$. According to the monotonicity of $g(u)$, we give a classification of growth rate per capita $g(u)$ on the first quadrant of $(q, r)$-parameter plane, see Figure 2. We have:

(i) If $(q, r) \in \Gamma_1 \cup \Gamma_3 \cup R_3 \cup R_5$, then $g(u) \leq 0$ for all $u \in [0, \infty)$. Hence problem (1.1) has no positive solution for all $\lambda > 0$.

We define the bifurcation curve of (1.1)

\[ \bar{S} = \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)} \}. \]

By (1.2), $g(0) = r - 1, g'(0) = m - r/q$, and $\lim_{u \to \infty} g(u) = -\infty$. According to the monotonicity of $g(u)$, we give a classification of growth rate per capita $g(u)$ on the first quadrant of $(q, r)$-parameter plane, see Figure 2. We have:

(i) If $(q, r) \in \Gamma_1 \cup \Gamma_3 \cup R_3 \cup R_5$, then $g(u) \leq 0$ for all $u \in [0, \infty)$. Hence problem (1.1) has no positive solution for all $\lambda > 0$.
(ii) If $(q, r) \in \Gamma_2 \cup R_4$, $g(u)$ has exactly one positive zero at some $\beta$, $g'(u) < 0$ on $(0, \beta)$, and $g(u) \leq 0$ on $[\beta, \infty)$. Thus, the bifurcation curve $\bar{S}$ of (1.1) is a monotone increasing curve since $f(u) - uf'(u) = -u^2g'(u) > 0$ on $(0, \beta)$, see Figures 3(iv) and 4.

(iii) If $(q,r) \in R_1$, $g(u)$ has exactly one positive zero at some $\beta$ such that $g(\beta) = 0$, $g(u) > 0$ on $(0, \beta)$, and $g(u) < 0$ on $(\beta, \infty)$. In addition, $g(u)$ changes from increasing to decreasing on $(0, \beta)$. In Theorem 2.1 stated below, we prove that the bifurcation curve $\bar{S}$ of (1.1) has exactly one critical point, where the curve turns to the right on the $(\lambda, \|u\|_\infty)$-plane when $(q, r) \in R_1$, see Figures 3(ii)-(iii) and 4.

(iv) If $(q, r) \in R_2$, $g(u)$ has exactly two positive zeros at some $\beta_1 < \beta$ such that $g(\beta_1) = g'(\beta) = 0$, $g(u) < 0$ on $(0, \beta_1) \cup (\beta, \infty)$ and $g(u) > 0$ on $(\beta_1, \beta)$. In addition, $g(u)$ changes from increasing to decreasing on $(0, \beta)$. Thus, for each fixed $q > 1/m$, we have

$$
\int_0^\beta f(u)du < 0 \quad \text{for } r \text{ near } r_1(q)^+,
$$

$$
\int_0^\beta f(u)du > 0 \quad \text{for } r \text{ near } 1^-.
$$

In addition, for each fixed $q > 1/m$,

$$
\frac{d}{dr} \int_0^\beta f(u)du = \frac{1}{6q} \beta^3 + \frac{1}{2r} \frac{\beta^2}{1 + m\beta + \beta^2} > 0
$$

Figure 4. Classified bifurcation curves of (1.1), drawn on the first quadrant of $(q, r)$-parameter plane.
because \( f(\beta) = 0 \). Hence for each fixed \( q > 1/m \), there exists \( \tilde{r}_1(q) \in (r_1(q), 1) \) such that

\[
\int_0^\beta f(u)du < 0, \quad \text{for} \quad r_1(q) < r < \tilde{r}_1(q),
\]

\[
\int_0^\beta f(u)du = 0, \quad \text{for} \quad r = \tilde{r}_1(q),
\]

\[
\int_0^\beta f(u)du > 0, \quad \text{for} \quad \tilde{r}_1(q) < r < 1.
\]

We define the curve

\[
\bar{\Gamma}_1 = \{(q, r) : q > \frac{1}{m} \text{ and } r = \tilde{r}_1(q)\}
\]

and regions

\[
\hat{R}_2 = \{(q, r) : 0 < r < mq \text{ and } (q, r) \text{ lies between the curve } \bar{\Gamma}_1 \text{ and the line } r = 1\},
\]

\[
\hat{R}_2 = \{(q, r) : 0 < r < mq \text{ and } (q, r) \text{ lies between curves } \Gamma_1 \text{ and } \bar{\Gamma}_1\}.
\]

(So \( R_2 = \hat{\Gamma}_1 \cup \hat{R}_2 \cup \hat{R}_2 \).) Notice that, for each fixed \((q, r) \in \hat{R}_2\), \( \int_0^\beta f(u)du > 0 \).

Thus, for each fixed \((q, r) \in \hat{R}_2\), there exists a positive number \( \gamma \in (\beta_1, \beta) \) satisfying \( \int_0^\gamma f(u)du = 0 \) since \( f(u) < 0 \) on \((0, \beta_1)\) and \( f(u) > 0 \) on \((\beta_1, \beta)\). In Theorem 2.2 stated below, we prove that the bifurcation curve \( \bar{S} \) of (1.1) is a C-shaped curve on the \((\lambda, \|u\|_\infty)\)-plane when \((q, r) \in \hat{R}_2\), see Figures 3(ii) and 4. In addition, we know that for \((q, r) \in \bar{\Gamma}_1 \cup \hat{R}_2\), the problem (1.1) has no positive solution for all \( \lambda > 0 \) since \( \int_0^\beta f(u)du \leq 0 \).

2. Main results

Let \( u_\lambda \) be a positive solution of (1.1) with \( \alpha = \|u_\lambda\|_\infty > 0 \).

**Theorem 2.1** (See Figures 3(ii)-(iii) and 4). Consider (1.1). If \((q, r) \in R_1\), then

\[
\lim_{\alpha \to 0^+} \lambda(\alpha) = \hat{\lambda} = \frac{\pi^2}{4(r-1)} \in (0, \infty], \quad \lim_{\alpha \to \beta^-} \lambda(\alpha) = \infty, \tag{2.1}
\]

and the bifurcation curve \( \bar{S} \) of (1.1) is a C-shaped curve on the \((\lambda, \|u\|_\infty)\)-plane. More precisely, \( \bar{S} \) consists of a continuous curve with exactly one turning point, \((\lambda_*, \|u_\lambda_\|_\infty)\) such that \( 0 < \lambda_* < \hat{\lambda} \leq \infty \) and \( 0 < \|u_\lambda_\|_\infty < \beta \), where the curve turns to the right. Problem (1.1) has:

(i) exactly two positive solutions \( u_\lambda, v_\lambda \) with \( u_\lambda < v_\lambda \) for \( \lambda_* < \lambda < \hat{\lambda} \),

(ii) exactly one positive solution \( u_\lambda \) for \( \lambda = \lambda_* \) and exactly one positive solution \( v_\lambda \) for \( \lambda \geq \hat{\lambda} \) (if \( r > 1 \)),

(iii) no positive solution for \( 0 < \lambda < \lambda_* \).

Furthermore,

\[
\lim_{\lambda \to \infty} \|u_\lambda\|_\infty = 0, \quad \text{if} \quad r = 1,
\]

\[
\lim_{\lambda \to (\lambda^-)} \|u_\lambda\|_\infty = 0, \quad \text{if} \quad r > 1, \tag{2.2}
\]

and

\[
\lim_{\lambda \to \infty} \|v_\lambda\|_\infty = \beta. \tag{2.3}
\]
The time map formula for studying problem (1.1) takes the form:

\[
T \text{ of the equation number of positive solutions of (1.1) is equivalent to studying the number of roots to the right. Problem m with Theorem 2.2, Lemma 3.2.}
\]

See Laetsch [8] for the derivation of the time map formula. So positive solutions

Furthermore, \((\bar{u}, \|u\|_\infty)\) consists of a continuous curve with exactly one turning point, \((\lambda_*, \|u_{\lambda_*}\|_\infty)\) such that \(0 < \lambda_* < \infty\) and \(\gamma < \|u_{\lambda_*}\|_\infty < \beta\), where the curve turns to the right. Problem (1.1) has:

(i) exactly two positive solutions \(u_\lambda, v_\lambda\) with \(u_\lambda < v_\lambda\) for \(\lambda_* < \lambda < \infty\),
(ii) exactly one positive solution \(u_\lambda\) for \(\lambda = \lambda_*\),
(iii) no positive solution for \(0 < \lambda < \lambda_*\).

Furthermore,

\[
\lim_{\lambda \to \infty} \|u_\lambda\|_\infty = \gamma, \quad \lim_{\lambda \to \infty} \|v_\lambda\|_\infty = \beta. \tag{2.5}
\]

3. PROOFS OF MAIN RESULTS

For \(f(u) = ug(u)\) from the analysis of \(g(u)\) in Section 1, we obtain the following result.

**Lemma 3.1.** Consider

\[
f(u) = ru(1 - \frac{u}{q}) - \frac{u}{1 + mu + u^2}
\]

with \(m \geq 1\), \(q, r > 0\).

(i) If \((q, r) \in R_1\), then there exists a positive number \(\beta\) such that \(f(0) = f(\beta) = 0\), \(f(u) > 0\) on \((0, \beta)\), and \(f(u) < 0\) on \((\beta, \infty)\).

(ii) If \((q, r) \in R_2\), then there exist two positive numbers \(\beta_1 < \beta\) such that \(f(0) = f(\beta_1) = f(\beta) = 0\), \(f(u) < 0\) on \((0, \beta_1) \cup (\beta, \infty)\), and \(f(u) > 0\) on \((\beta_1, \beta)\).

Also, there exists a positive number \(\gamma \in (\beta_1, \beta)\) satisfying \(\int_0^\gamma f(u)du = 0\).

Let \(F(u) \equiv \int_0^u f(t)dt\), and \(u_\alpha\) be a positive solution of (1.1) with \(\alpha \equiv \|u_\lambda\|_\infty > 0\).

The time map formula for studying problem (1.1) takes the form:

(i) if \((q, r) \in R_1\), then the time map is

\[
T(\alpha) = \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{|F(\alpha) - F(u)|^{1/2}}du = \sqrt{\lambda} \quad \text{for} \quad 0 < \alpha < \beta; \tag{3.1}
\]

(ii) if \((q, r) \in R_2\), then the time map is

\[
T(\alpha) = \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{|F(\alpha) - F(u)|^{1/2}}du = \sqrt{\lambda} \quad \text{for} \quad \gamma < \alpha < \beta. \tag{3.2}
\]

See Laetsch [8] for the derivation of the time map formula. So positive solutions \(u_\lambda\) of (1.1) correspond to \(\|u_\lambda\|_\infty = \alpha\) and \(T(\alpha) = \sqrt{\lambda}\). Thus, studying the exact number of positive solutions of (1.1) is equivalent to studying the number of roots of the equation \(T(\alpha) = \sqrt{\lambda}\). We define

\[
\theta(u) = 2F(u) - uf(u) = \frac{r}{3q} u^3 + \frac{u^2}{1 + mu + u^2} - 2\int_0^u \frac{t}{1 + mt + t^2}dt. \tag{3.3}
\]

**Lemma 3.2.** Consider

\[
f(u) = ru(1 - \frac{u}{q}) - \frac{u}{1 + mu + u^2}
\]
with \( m \geq 1, q, r > 0 \). If \((q, r) \in R_1 \cup \hat{R}_2\), then there exists a positive number \( B \in (0, \beta) \) such that

\[
\theta''(u) = -uf''(u) \begin{cases} 
= 0 & \text{for } u = B, \\
< 0 & \text{on } (0, B), \\
> 0 & \text{on } (B, \beta).
\end{cases}
\]

**Proof.** For \( m \geq 1 \) and \( q, r > 0 \), by (3.3), we have

\[
\theta'(u) = f(u) - uf'(u) = u^2 \left[ \frac{r}{q} - \frac{m + 2u}{(1 + mu + u^2)^2} \right],
\]

\[
\theta''(u) = -uf''(u) = 2u \left[ \frac{r}{q} - \frac{m + 3u - u^3}{(1 + mu + u^2)^2} \right] = 2u \left[ \frac{r}{q} - I(u) \right],
\]

where

\[ I(u) = \frac{m + 3u - u^3}{(1 + mu + u^2)^3}. \]

We compute that

\[ I'(u) = \frac{3u^4 - 18u^2 - 12mu + 3(1 - m^2)}{(1 + mu + u^2)^4}. \]

Now we define

\[ p(u) = 3u^4 - 18u^2 - 12mu + 3(1 - m^2) \]

and obtain

\[ p'(u) = 12(u^3 - 3u - m), \quad p''(u) = 36(u^2 - 1). \]

Therefore,

\[
p''(1) = 0, \\
p''(u) < 0 \text{ on } [0, 1), \\
p''(u) > 0 \text{ on } (1, \infty).
\]

By (3.4), there exists \( C > 0 \) such that

\[ p(C) = 0, \]

\[ p(u) < 0 \text{ on } (0, C), \]

\[ p(u) > 0 \text{ on } (C, \infty). \]

Then

\[
I'(C) = 0, \\
I'(u) < 0 \text{ on } (0, C), \\
I'(u) > 0 \text{ on } (C, \infty).
\]

By (3.5), \( I(0) = m \geq 1 \) and \( \lim_{u \to \infty} I(u) = 0 \), there exists a \( D \) with \( 0 < D < C \) such that

\[ I(D) = 0, \]

\[ I(u) > 0 \text{ on } (0, D), \]

\[ I(u) < 0 \text{ on } (D, \infty). \]
It follows that \( \max_{u \in [0, \infty)} I(u) = I(0) = m \) and \( I(u) \) is strictly decreasing on \([0, D]\). So we obtain that for \( 0 < r/q \leq m \), there exists \( B > 0 \) such that \( I(B) = r/q \) and
\[
\begin{align*}
\theta''(B) &= 0, \\
\theta''(u) &= 0 \quad \text{on } (0, B), \\
\theta''(u) &= 0 \quad \text{on } (B, \infty).
\end{align*}
\]
In addition, if \( 0 < r/q \leq m \), we have
\[
\theta'(u) = u^2 [ \frac{r}{q} - \frac{m + 2u}{1 + mu + u^2} ] > 0 \quad \text{for } u \text{ large enough.}
\]
By \( \theta(0) = \theta'(0) = 0 \), there exists \( E > B \) such that \( \theta(0) = \theta(E) = 0 \), \( \theta(u) < 0 \) on \((0, E)\), \( \theta(u) > 0 \) on \((E, \infty)\).

(i) For \((q, r) \in R_1\), we know that \( 0 < r/q < m \) and \( \int_0^\beta f(u)du > 0 \) by \( f(\beta) = 0 \). It follows that \( \beta > E > B \) by \( \theta''(u) = -uf''(u) \). So we obtain \( B \in (0, \beta) \) and
\[
\theta''(u) = -uf''(u) \begin{cases} 
= 0 & \text{for } u = B, \\
< 0 & \text{on } (0, B), \\
> 0 & \text{on } (B, \beta)
\end{cases}
\]
by \( \theta''(u) = -uf''(u) \). The proof of Lemma 3.2 is complete.

(ii) For \((q, r) \in \hat{R}_2\), we know that \( 0 < r/q < m \) and \( \int_0^\beta f(u)du > 0 \). Thus,
\[
\theta(\beta) = 2F(\beta) - \beta f(\beta) = 2 \int_0^\beta f(u)du > 0
\]
by \( f(\beta) = 0 \). It follows that \( \beta > E > B \) by \( \theta''(u) = -uf''(u) \). So we obtain \( B \in (0, \beta) \) and
\[
\theta''(u) = -uf''(u) \begin{cases} 
= 0 & \text{for } u = B, \\
< 0 & \text{on } (0, B), \\
> 0 & \text{on } (B, \beta)
\end{cases}
\]
by \( \theta''(u) = -uf''(u) \). The proof of Lemma 3.2 is complete.

By Lemmas 3.1 (i), 3.2 and a slight modification of the proof of [22] Theorem, we obtain the following Lemma.

**Lemma 3.3.** Consider \((1.1)\) with \( m \geq 1, q, r > 0 \). If \((q, r) \in R_1\), then
\[
\lim_{\alpha \to 0^+} T(\alpha) = \begin{cases} 
\frac{\pi}{2\sqrt{f'(0)}} \in (0, \infty), & \text{if } f'(0) > 0, \\
\infty, & \text{if } f'(0) = 0,
\end{cases}
\]
\[
\lim_{\alpha \to \beta^-} T(\alpha) = \infty.
\]
In addition, \( T(\alpha) \) has exactly one critical point, a minimum, on \((0, \beta)\).
Proof of Theorem 2.1. By (3.1), \( f'(0) = r - 1 \) and Lemma 3.3, the results in (2.1) hold, and the bifurcation curve \( \bar{S} \) of (1.1) is a C-shaped curve on the \((\lambda, \|u\|_\infty)\)-plane. More precisely, \( \bar{S} \) consists of a continuous curve with exactly one turning point, \((\lambda_*, \|u_{\lambda_*}\|_\infty)\) such that \( 0 < \lambda_* < \hat{\lambda} \leq \infty \) and \( 0 < \|u_{\lambda_*}\|_\infty < \beta \), where the curve turns to the right. So we obtain immediately the exact multiplicity result and ordering results of the solutions in parts (i)–(iii). The proofs of results in (2.2) and (2.3) are easy but tedious; we omit them.

By Lemmas 3.1 (ii), 3.2, and [23, Theorem 1, Notes 1, 2 and Remark 2], we obtain the following Lemma.

Lemma 3.4. Consider (1.1) with \( m \geq 1, q, r > 0 \). If \((g, r) \in \tilde{R}_2\), then
\[
\lim_{\alpha \to \gamma^+} T(\alpha) = \lim_{\alpha \to \beta^-} T(\alpha) = \infty.
\]
In addition, \( T(\alpha) \) has exactly one critical point, a minimum, on \((\gamma, \beta)\).

Proof of Theorem 2.2. By (3.2) and Lemma 3.4, the results in (2.4) hold, and the bifurcation curve \( \bar{S} \) of (1.1) is a C-shaped curve on the \((\lambda, \|u\|_\infty)\)-plane. More precisely, \( \bar{S} \) consists of a continuous curve with exactly one turning point, \((\lambda_*, \|u_{\lambda_*}\|_\infty)\) such that \( 0 < \lambda_* < \infty \) and \( \gamma < \|u_{\lambda_*}\|_\infty < \beta \), where the curve turns to the right. So we obtain immediately the exact multiplicity result and ordering results of the solutions in parts (i)–(iii). The proofs of results in (2.5) are easy but tedious; we omit them.

Acknowledgments. T.-S. Yeh was supported by the Ministry of Science and Technology of the republic of China.

References

**Jyun-Yuan Ciou**
Department of Applied Mathematics, National University of Tainan, Tainan 700, Taiwan ROC

*Email address: b7626529@gmail.com*

**Tzung-Shin Yeh**
Department of Applied Mathematics, National University of Tainan, Tainan 700, Taiwan, ROC

*Email address: tsyeh@mail.nutn.edu.tw*