EXTINCTION IN FINITE TIME OF SOLUTIONS TO FRACTIONAL PARABOLIC POROUS MEDIUM EQUATIONS WITH STRONG ABSORPTION

NGUYEN ANH DAO

Dedicated to Prof. Jesus Ildefonso Díaz on his 70th birthday

ABSTRACT. In this article we study the solutions of a general fractional parabolic porous medium equation with a non-Lipschitz absorption term. We obtain the existence of weak solutions, $L^p$-estimates, and decay estimates. Also, we show that weak solutions must vanish after a finite time, even for large initial data.

1. Introduction

In this article, we study the fractional parabolic porous medium equation with a non-Lipschitz absorption term,

$$\partial_t u - \text{div}(|u|^{m_1} \nabla (-\Delta)^{-s}|u|^{m_2-1}u) + |u|^\beta u = 0 \text{ in } \mathbb{R}^N \times (0, T),$$

$$u(x, 0) = u_0(x) \text{ in } \mathbb{R}^N,$$

where $m_1, m_2 > 0$, $s \in (0, 1)$, $\beta \in (0, 1)$, and $N \geq 2$. Equations of type (1.1) with $s = 0$ and $m_2 = 1$, without the absorption term, correspond to the well-known porous medium equation $\partial_t u = \text{div}(u^{m_1} \nabla u)$.

This equation appears in applications such as the standard model for gas flow through a porous medium (Darcy-Leibenzon-Muskat), Boussinesq’s model of groundwater flow, and a model of population dynamics (Gurtin-McCamy) (see [19]). These applications have served as a motivation for many authors to study equation (1.1), see for example [2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 16, 17], and the references therein. Most of the known results concern the existence of weak solutions, decay estimates and finite speed of propagation. This is the main feature of porous media equations and gave rise to free boundary problems.

Now, we like to mention some recent results concerning equation (1.1). Biler et al. [1] studied (1.1) with $\alpha = 2(1-s)$, $m_1 = 1$, $m = m_2 + 1$, without the absorption term $|u|^\beta u$:

$$\partial_t u - \text{div}(|u|^{\alpha-1}(|u|^{m_2-2}u)) = 0,$$

2010 Mathematics Subject Classification. 35R11, 35K65.
Key words and phrases. Nonlocal nonlinear parabolic equation; fractional Laplacian; finite time extinction.
©2021 Texas State University.
They constructed nonnegative self-similar solutions of Barenblatt-Pattle-Zeldovich type, and obtained an existence of weak solutions $u$ satisfying the decay estimate
\[ \|u(t)\|_{L^p} \leq Ct^{-\frac{N(1-p)}{m-1+p+\alpha}} \|u_0\|_{L^1}^{\frac{N(m-1)p+\alpha}{N(m-1)+p+\alpha}}. \]
(1.2)
Stan et al. \[18\] investigated (1.1) with $u$ type, and obtained an existence of weak solutions without the absorption term. The authors studied the existence of nonnegative weak solutions for all integrable initial data $u_0$. They obtained the smoothing effect $L^p$-$L^\infty$, for $p \geq 1$:
\[ \|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{m-1+2p(1-s)}} \|u_0\|_{L^p}^{\frac{2p(1-s)}{N(m-1)+2p(1-s)}}, \]
(1.3)
with $C = C(N, s, m, p) > 0$. Moreover, the finite and infinite speed of propagation have been also studied by the same authors in \[17\]. Very recently, Dao-Díaz studied (1.1), and obtained the following result.

**Theorem 1.1** (\[12\]). Let $m_1, m_2 > 0$ and $s \in (0, 1)$. Suppose that $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then, there exists a weak solution $u$ of (1.1) satisfying the following properties:

(i) $L^q$-estimates: For any $1 \leq q \leq \infty$, we have
\[ \|u(t)\|_{L^q} \leq \|u_0\|_{L^q}, \quad \text{for a.e. } t \in (0, T). \] (1.4)

(ii) Decay estimates: Let $p \geq 1$ be such that $m_1 + m_2 > 1 - \frac{2p(1-s)}{N}$. Then
\[ \|u(t)\|_{L^\infty} \leq Ct^{-\frac{1}{p(m_1+m_2)-\sigma_0}} \|u_0\|_{L^p}^{\frac{p(1-\alpha_0)}{p(m_1+m_2)-\sigma_0}}, \] (1.5)
with $\alpha_0 = (N - 2(1-s))/N$, and $\sigma_0 = m_1 + m_2 - 1$.

(iii) Finite time extinction: If $m_1 + m_2 < \alpha_0$ then, there is a finite time $T_0 > 0$ such that
\[ u(x, t) = 0, \quad \text{for } (x, t) \in \mathbb{R}^N \times (T_0, \infty). \] (1.6)

Inspired by the above results, we want to prove the existence of weak solutions to equation (1.1), which satisfies estimates (1.4), (1.5). After that, we show that such a weak solution must vanish after a finite time, even when beginning with a large initial data $u_0$. It is known that this phenomenon occurs because of the strong absorption term $|u|^{\beta-1}u$. see \[11\] \[13\] \[14\] for another strong absorption term $u^{-\beta}$. Let us define
\[ \Theta(u) = |u|^{m_1} \nabla(-\Delta)^{-s}|u|^{m_2-1}, \quad Q_T = \mathbb{R}^N \times (0, T). \]

**Definition 1.2.** Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We say that $u$ is a weak solution of (1.1) if $u \in L^1(0, T; L^\infty(\mathbb{R}^N)) \cap L^\infty(Q_T)$ satisfies $\text{div} \Theta(u) \in L^2(0, T; Y(B_R))$, and
\[ \int_0^T \int_{\mathbb{R}^N} (-u \varphi_t + \Theta(u) \cdot \nabla \varphi + |u|^{\beta-1} u \varphi) \, dx \, dt = 0, \quad \forall \varphi \in C_c^\infty(Q_T), \]
where
\[ Y(B_R) = \begin{cases} H^{-1}(B_R), & \text{if } s \in [1/2, 1), \\ W^{-2p}(B_R), & \text{if } s \in (0, 1/2). \end{cases} \]
Note that $H^{-1}(B_R)$ is the dual space of $H^1_0(B_R)$, and $W^{-2p}(B_R)$ the dual space of $W_0^{2,p}(B_R)$. Here $B_R$ is the ball in $\mathbb{R}^N$, with center at 0 and radius $R$. 
Remark 1.3. It follows from Definition 1.2 that $u \in C([0, T]; Y(B_R))$, for any $R > 0$. Thus, $u(t)$ possesses an initial trace $u_0$ in this sense. In particular, if either $s \in (1/2, 1)$ or $m_2 > m_1$, then $u \in C([0, T]; H^{-1}(B_R))$ for every $R > 0$.

Our main results read as follows.

Theorem 1.4. Let $s \in (0, 1)$, $\beta \in (0, 1)$, and $m_1, m_2 > 0$. Suppose that $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then, there exists a weak solution of (1.1) satisfying (1.4), (1.5), and (1.6) in Theorem 1.1.

Concerning the finite time extinction of solutions, it suffices to consider $m_1 + m_2 \geq \alpha_0$ in the following theorem since $u$ vanishes after a finite time if provided $m_1 + m_2 < \alpha_0$.

Theorem 1.5. Assume the hypotheses in Theorem 1.4. Suppose that $m_1 + m_2 \geq \alpha_0$. Then, there exists a finite time $T_0 > 0$ such that

$$u(x, t) = 0, \quad \text{for } (x, t) \in \mathbb{R}^N \times (T_0, \infty).$$

(1.7)

And $T_0$ can be estimated as follows

$$T_0 \leq C\|u_0\|^p_{L^p(1-\gamma_0)},$$

(1.8)

for some constant $C > 0$ (independent of $u_0$), with

$$\gamma_0 = \frac{1}{1 + 2(1-s)p^{-1}N(m_1 + m_2) - \beta(N - 2(1-s))}.$$

Note that $\gamma_0 \in (0, 1)$ since $m_1 + m_2 \geq \alpha_0$.

Through this paper, the constant $C$ may change value from step by step. Moreover, $C = C(\alpha, \beta, \gamma)$ means that the constant $C$ merely depends on the parameters $\alpha, \beta, \gamma$. We denote $\| \cdot \|_{X(\mathbb{R}^N)} = \| \cdot \|_{X}$, and $\int_{\mathbb{R}^N} f(x) dx = \int f(x) dx$. Finally, $A \lesssim B$ means that there exists a positive constant $c$, independent of the data, such that $A \leq cB$.

2. Functional setting

Let $p \geq 1$, and $s \in (0, 1)$. For a given domain $\Omega \subset \mathbb{R}^N$, we define the fractional Sobolev space

$$W^{s,p}(\Omega) = \{ u \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy < \infty \},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left( \|u\|_{L^p(\Omega)}^p + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{1/p}.$$

We also denote the homogeneous fractional Sobolev space by $\dot{W}^{s,p}(\Omega)$, endowed with the seminorm

$$\|u\|_{\dot{W}^{s,p}(\Omega)} = \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{1/p}.$$

In particular, we denote $W^{s,2}(\mathbb{R}^N)$ by $H^s(\mathbb{R}^N)$, which turns out to be a Hilbert space. It is well-known that we have the equivalent characterization

$$H^s(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : \int (1 + |\xi|^{2s})|\mathcal{F}(u)(\xi)|^2 d\xi < \infty \},$$
Lemma 2.2. Let $\psi \in C^1(\mathbb{R})$ be such that $\psi'$, $\psi'' \geq 0$. Then

$$\int \psi(f) L_\varepsilon^s[f] \, dx \geq 0. \quad (2.4)$$

If we take $\psi(f) = f$, then we obtain

$$\int f L_\varepsilon^s[f] \, dx \geq \int |L_\varepsilon^{s/2} \Phi(f)|^2 \, dx, \quad (2.5)$$

where $\Phi' = (\Phi')^2$.

Finally, we have the following fundamental inequality.
Lemma 2.3. Let $\alpha, \beta > 0$, and $\theta = \frac{\alpha + \beta}{2}$. Then, there is a constant $C > 0$ such that

$$||a|^{\theta-1}a - |b|^{\theta-1}b||^2 \leq C ||a|^{\alpha-1}a - |b|^{\alpha-1}b|| ||a|^{\beta-1}a - |b|^{\beta-1}b||,$$ \quad $\forall a, b \in \mathbb{R}$. (2.6)

3. Existence of solutions

In this section, we prove Theorem 1.4 using the Lemmas and the Propositions below.

A regularized problem. We consider the following regularizing version of (1.1),

$$\partial_t u - \delta_1 \Delta u + \delta_2 L_\varepsilon^\alpha [J_\kappa (u)] - \text{div} \Theta_{\varepsilon, \mu} (u) + |u|^{\beta-1} u \Phi_\mu (u) = 0,$$

in $\mathbb{R}^N \times (0, T)$,

$$u(0) = u_0, \quad \text{in } \mathbb{R}^N,$$

where $s_0 = (1 - 2s)_+$, $\Theta_{\varepsilon, \mu} (u) = H_\varepsilon (u) \nabla (-\Delta)^{-1} L_\varepsilon^{1-s} [G_\mu (u)]$, and

$$H_\varepsilon (u) = \frac{|u|m_{1+2}}{\nu^2 + u^2}, \quad G_\mu (u) = \frac{|u|m_{2+1}}{\nu^2 + u^2}, \quad J_\kappa (u) = \frac{|u|m_0+1}{\nu^2 + \kappa^2},$$

with $m_0 = \frac{1}{2} \min \{ m_1, \frac{m_2(N-2s)}{N} \}$, and $\Phi_\mu (u)$ is a cut-off function in a neighborhood of $u = 0$, defined by $\Phi_\mu (u) = \Phi (\frac{u}{\mu})$, where $\Phi (s) \in \mathcal{C}^\infty (\mathbb{R})$, $0 \leq \Phi (s) \leq 1$, for all $s \in \mathbb{R}$, and

$$\Phi (s) = \begin{cases} 0, & \text{if } |s| \leq 1, \\ 1, & \text{if } |s| \geq 2, \end{cases}$$

for $\delta_1, \delta_2, \varepsilon, \kappa, \mu, \nu \in (0, 1)$.

We shall prove the existence of solutions of (3.1) in a suitable functional space by using the fixed-point theorem, and derive some energy estimates in order to pass to the limit as $\varepsilon, \kappa, \nu, \delta_1, \delta_2, \mu \to 0$ alternatively. The proof is most likely to the one in Section 3, [12]. Here, we just present some different points with the presence of the absorption.

Let us put

$$X = L^1 (\mathbb{R}^N) \cap L^\infty (\mathbb{R}^N),$$

with the associated norm $\| \cdot \|_X = \| \cdot \|_{L^1 (\mathbb{R}^N)} + \| \cdot \|_{L^\infty (\mathbb{R}^N)}$.

Lemma 3.1. Let $u_0 \in X$ and $f \in L^1 (Q_T) \cap L^\infty (Q_T)$. Then, there exists a weak solution $u \in C ([0, T]; X)$ satisfying problem (3.1) in the weak sense, i.e.

$$\int_0^T \int ( - u \varphi_t + \delta_1 \nabla u \cdot \nabla \varphi + \delta_2 L_\varepsilon^\alpha [J_\kappa (u)] \varphi - \Theta_{\varepsilon, \mu} (u) \cdot \nabla \varphi + |u|^{\beta-1} u \Phi_\mu (u) \varphi ) \, dx \, dt = 0,$$

for all $\varphi \in C_c^\infty (Q_T)$.

Proof. We look for a mild solution $u \in C ([0, T]; X)$ as a fixed point of the map

$$T (u) = e^{b_1 \Delta} u_0 + \int_0^t \nabla e^{(t-\tau) b_1 \Delta} \Theta_{\varepsilon, \mu} (u) \, d\tau - \int_0^t e^{(t-\tau) \delta_2 L_\varepsilon^\alpha [J_\kappa (u)] + |u|^{\beta-1} u \Phi_\mu (u)} \, d\tau,$$

where $\exp$ is the semigroup corresponding to the heat kernel $(4\pi t)^{-N/2} \exp (-|x|^2/4t)$.
We note that $|u|^{\beta-1}u \Phi_\mu(u)$ is a locally Lipschitz function, then we can mimic the proof of \cite[Theorem 4]{[12]} to obtain that $T$ maps $C([0,T];X)$ into itself. Moreover, there is a real number $\gamma \in (0,1)$ such that
\[
\|T(u) - T(v)\|_{C([0,T];X)} \leq C(R)T^\gamma\|u - v\|_{C([0,T];X)},
\]
for all $u, v \in B(0,R) \subset C([0,T];X)$. This implies that
\[
\|T(u) - T(v)\|_{C([0,T];X)} \leq \frac{1}{2}\|u - v\|_{C([0,T];X)},
\]
if $T > 0$ is chosen small enough. Thanks to the contraction mapping theorem, we obtain a unique mild solution $u$ to equation $T(u) = u$.

Finally, since the terms in (3.1) are regular, then it follows from the standard regularity theory that $u$ is smooth in $\mathbb{R}^N \times (0,T)$. The proof is complete. \hfill $\square$

Now, we prove an $L^q$-estimate of $u$.

**Proposition 3.2.** Let $u$ be a solution of (3.1) in $Q_T$. Then, for every $q \in [1, \infty]$, we have
\[
\|u(t)\|_{L^q(\mathbb{R}^N)} \leq \|u_0\|_{L^q(\mathbb{R}^N)}, \quad \forall t \in (0,T).
\]}

**Proof.** For every $q > 1$, by testing $|u|^{q-2}u$ to (3.1), we obtain
\[
\frac{1}{q} \frac{d}{dt} \int |u(t)|^q \, dx + (q-1) \int |u|^{q-2}H_\nu(u)\nabla(-\Delta)^{-1}L^{1-s}_\varepsilon[G_\nu(u)] \cdot \nabla u \, dx
\]
\[
+ \delta_1(q-1) \int |u|^{q-2}\nabla u^2 \, dx + \delta_2 \int L^{q_0}_\varepsilon[J_\kappa(u)]|u|^{q-2}u \, dx
\]
\[
+ \int |u|^{\beta+q-1} \Phi_\mu(u) \, dx = 0,
\]
By applying Lemma \ref{Lemma 2.2} to $\psi(u) = |u|^{q-2}u$, and $\phi(u) = J_\kappa(u)$, we obtain
\[
\int |u|^{q-2}uL^{q_0}_\varepsilon[J_\kappa(u)] \, dx \geq 0.
\]}

Next, we observe that
\[
\int |u|^{q-2}H_\nu(u)\nabla(-\Delta)^{-1}L^{1-s}_\varepsilon[G_\nu(u)] \cdot \nabla u \, dx
\]
\[
= \int \nabla(-\Delta)^{-1}L^{1-s}_\varepsilon[G_\nu(u)] \cdot \nabla \tilde{H}_\nu(u) \, dx
\]
\[
= \int \tilde{H}_\nu(u)(-\Delta)(-\Delta)^{-1}L^{1-s}_\varepsilon[G_\nu(u)] \, dx
\]
\[
= \int \tilde{H}_\nu(u)L^{1-s}_\varepsilon[G_\nu(u)] \, dx \geq 0,
\]
with
\[
\tilde{H}_\nu(u) = \int_0^u |s|^{q-2}H_\nu(s) \, ds.
\]
Note that the inequality in (3.5) was also obtained by Lemma \ref{Lemma 2.2}. Thus,
\[
\frac{d}{dt} \int |u(t)|^q \, dx \leq 0.
\]
This implies (3.2) for any $q \in [1, \infty)$. Finally, passing to the limit as $q \to \infty$, we also obtain (3.2) for the $L^\infty$-estimate.
It remains to prove the $L^1$-estimate of $u$. For every $\eta > 0$, let us put
\[
\chi_\eta(r) = \begin{cases} 
\text{sign}(r), & \text{if } |r| > \eta, \\
 r/\eta, & \text{if } |r| \leq \eta,
\end{cases}
\]
Testing (3.1) with $\chi_\eta(u)$ yields
\[
\int (u_t \chi_\eta(u) + \delta_1 \nabla u \cdot \nabla \chi_\eta(u) + \delta_2 \mathcal{L}_\varepsilon^{0\eta}(J_\kappa(u))\chi_\eta(u) + \Theta(u) \cdot \nabla \chi_\eta(u)) \, dx \\
+ \int |u|^{\beta-1} u \Phi_{\mu}(u) \chi_\eta(u) \, dx = 0. 
\] (3.6)

Since $\chi'_\eta(u) \geq 0$, it is clear that
\[
\int \nabla u \cdot \nabla \chi_\eta(u) \, dx = \int |\nabla u|^2 \chi'_\eta(u) \, dx \geq 0,
\]
and by Lemma 2.2 we have
\[
\int \mathcal{L}_\varepsilon^{0\eta}(J_\kappa(u))\chi_\eta(u) \, dx \geq 0, \quad \int \Theta(u) \cdot \nabla \chi_\eta(u) \, dx \geq 0.
\]

From (3.6) after integrating on $(0, t)$ it follows that
\[
\int S_\eta(u(t)) \, dx \leq \int S_\eta(u_0) \, dx,
\]
with
\[
S_\eta(u) = \int_0^u \chi_\eta(r) \, dr = \frac{u^2}{2\eta} \chi_{\{|u|<\eta\}} + (|u| - \frac{\eta}{2}) \chi_{\{|u|\geq\eta\}},
\]
where $\chi_A$ denotes the characteristic function of the set $A$. Note that
\[
\lim_{\eta \to 0} \int S_\eta(u(t)) \, dx = \int |u(t)| \, dx.
\]
So, (3.2) follows with $q = 1$. This completes the proof. \qed

The following results are similar to the ones in [12], so we omit their proofs.

**Proposition 3.3.** Let $u$ be as in Proposition 3.2. Then, there is a constant $C = C(m_0, u_0) > 0$ such that for every $\kappa, \varepsilon, \mu, \nu > 0$,
\[
\delta_2 \| \mathcal{L}_\varepsilon^{0\eta}(J_\kappa(u_\varepsilon)) \|_{L^2(Q_T)} \leq C. 
\] (3.7)

**Limit as $\varepsilon \to 0$.**

**Proposition 3.4.** Let $u_\varepsilon$ be the solution of problem (3.1). Then, there exists a subsequence of $\{u_\varepsilon\}_{\varepsilon > 0}$ (still denoted as $\{u_\varepsilon\}_{\varepsilon > 0}$) such that for any $R > 0$,
\[
u \to u_\varepsilon \quad \text{in } L^2(B_R \times (0, T)).
\]
Moreover, $u \in L^\infty(0, T; L^1(\mathbb{R}^N) \cap L^\infty(Q_T) \cap L^2(0, T; H^1(\mathbb{R}^N)))$ is a solution of the problem
\[
\begin{aligned}
  u_t - \delta_1 \Delta u - \text{div}(H_{\nu}(u) \nabla (-\Delta)^{-s} [G_{\nu}(u)]) + \delta_2 (-\Delta)^s J_\kappa(u) + |u|^{\beta-1} u \Phi_{\mu}(u) \\
  = 0, \quad \text{in } Q_T.
\end{aligned}
\] (3.8)
3.1. Limit as $\kappa \to 0$.

**Proposition 3.5.** Let $u_\kappa$ be the solution of problem (3.8). Then, for any $R > 0$ it holds
\[
u \to u, \quad \text{in } L^2(B_R \times (0, T))
\]
up to a subsequence. Moreover, $u \in L^\infty(0, T; L^1(\mathbb{R}^N)) \cap L^\infty(Q_T) \cap L^2(0, T; H^1(\mathbb{R}^N))$ is a solution of the problem
\[
u - \delta_1 \Delta u - \text{div} \Theta(v) + \delta_2 (-\Delta)^s |u|^{m_0-1} u + |u|^{\beta-1} u \Phi(u) = 0, \quad \text{in } Q_T, \tag{3.9}
\]
where we denote $\Theta(v) = H_v(u) \nabla (-\Delta)^{-s}[G_v(u)]$.

3.2. Limit as $\nu \to 0$.

**Proposition 3.6.** Let $u_\nu$ be the solution, obtained in Proposition 3.5. Then, there exists a subsequence of $\{u_\nu\}_{\nu > 0}$ converging to a function $u$ in $L^2(B_R \times (0, T))$ for any $R > 0$. Moreover, $u \in L^\infty(0, T; L^1(\mathbb{R}^N)) \cap L^\infty(Q_T) \cap L^2(0, T; H^1(\mathbb{R}^N))$ is a solution of the equation
\[
u - \delta_1 \Delta u - \text{div} \Theta(u) + \delta_2 (-\Delta)^s |u|^{m_0-1} u + |u|^{\beta-1} u \Phi(u) = 0, \quad \text{in } Q_T. \tag{3.10}
\]
Recall that $\Theta(u) = H(u) \nabla (-\Delta)^{-s}[G(u)]$, with $H(u) = |u|^{m_1}$ and $G(u) = |u|^{m_2-1} u$. 

3.3. Limit as $\delta_1, \delta_2 \to 0$.

**Proposition 3.7.** Let $u_{\delta_2}$ be a solution of (3.10) above. Then, there exists a subsequence of $\{u_{\delta_2}\}_{\delta_2 > 0}$ converging to a function $u$ in $L^2(B_R \times (0, T))$ for any $R > 0$. Moreover, $u \in L^\infty(0, T; L^1(\mathbb{R}^N)) \cap L^\infty(Q_T) \cap L^2(0, T; H^1(\mathbb{R}^N))$ is a weak solution of the problem
\[
u - \delta_1 \Delta u - \text{div} \Theta(u) + |u|^{\beta-1} u \Phi(u) = 0, \quad \text{in } Q_T. \tag{3.11}
\]

We emphasize that the estimates in the proof of Proposition 3.7 are also independent of $\delta_1$.

**Proposition 3.8.** Let $u_{\delta_1}$ be a solution of (3.11). Then there exists a subsequence of $\{u_{\delta_1}\}_{\delta_1 > 0}$, converging to a function $u$ in $L^2(B_R \times (0, T))$ for any $R > 0$. Furthermore, $u \in L^\infty(0, T; L^1(\mathbb{R}^N)) \cap L^\infty(Q_T)$, which is a weak solution of the equation
\[
u - \text{div} \Theta(u) + |u|^{\beta-1} u \Phi(u) = 0, \quad \text{in } Q_T. \tag{3.12}
\]
In addition, $\text{div}(\Theta(u))$ satisfies the following regularity:

- If $s \in [1/2, 1)$, then
\[
\text{div}(\Theta(u)) \in L^2(0, T; H^{-1}(B_R)). \tag{3.13}
\]

- If $s \in (0, 1/2)$, then
\[
\text{div}(\Theta(u)) \in L^p(0, T; W^{-2,p}(\mathbb{R}^N)), \tag{3.14}
\]
for $p > 1$ such that $\frac{mp}{p+1} \geq 1$, and $W^{-2,p}(\mathbb{R}^N)$ is the dual space of $W^{2,p}(\mathbb{R}^N)$.  

Let \( \mu \to 0 \).

**Proposition 3.9.** Let \( u_\mu \) be a solution of (3.12). Then, there exists a subsequence of \( \{ u_\mu \}_{\mu > 0} \), converging to a function \( u \) in \( L^2(\mathbb{B}_R \times (0, T)) \) for any \( R > 0 \). Furthermore, \( u \in L^\infty(0, T; L^1(\mathbb{R}^N)) \cap L^\infty(Q_T) \), which is a weak solution of equation (1.1). Also \( \text{div}(\Theta(u)) \) satisfies either (3.13) if \( s \in [1/2, 1) \), or (3.14) if \( s \in (0, 1/2) \).

Then, it is clear that solution \( u \), obtained from Proposition 3.9 is a weak solution of (1.1). Moreover, \( u \) also satisfies the energy inequality

\[
\frac{1}{p} \frac{d}{dt} \int |u(x, t)|^p \, dx + (p - 1) \int \left( \frac{G(u(x)) - G(u(y))}{|x - y|^{N+2(1-s)}} \right) |u|^{m_1+p-2}u(x) - |u|^{m_1+p-2}u(y) \, dx \, dy \leq 0 ,
\]

see (4.1) below. Thus, we can mimic the proof of [12, Theorem 2] to obtain decay estimate (1.5). This completes the proof of Theorem 1.4.

4. Finite time extinction of solutions

**Proof of Theorem 1.5.** For every \( p > 1 \), it follows from (3.3) that

\[
\frac{1}{p} \frac{d}{dt} \int |u(x, t)|^p \, dx + \int |u(x, t)|^{p-1+\beta} \, dx + (p - 1) \int \left( \frac{G(u(x)) - G(u(y))}{|x - y|^{N+2(1-s)}} \right) |u|^{m_1+p-2}u(x) - |u|^{m_1+p-2}u(y) \, dx \, dy \leq 0 .
\]

(4.1)

Thanks to Lemma 2.3, we obtain

\[
\frac{1}{p} \frac{d}{dt} \int |u(x, t)|^p \, dx + \int |u(x, t)|^{p-1+\beta} \, dx + C(p - 1) \int \left( \frac{|u|^{\theta_0-1}u(x) - |u|^{\theta_0-1}u(y)}{|x - y|^{N+2(1-s)}} \right)^2 \, dx \, dy \, dt \leq 0 ,
\]

with \( \theta_0 = (m_1 + m_2 + p - 1)/2 \). Next, applying the Sobolev embedding yields

\[
\| u(t) \|^{\theta_0}_{L^{2^*}} \leq C \| u(t) \|^{\theta_0}_{H^{1-s}} ,
\]

with \( C = C(N, s) \), and \( 2^* = \frac{2N}{N-2(1-s)} = \frac{2}{\alpha_0} \). Combining these inequalities yields

\[
\frac{1}{p} \frac{d}{dt} \| u(t) \|_L^p + C(\| u(t) \|_{L^{p_0-1+\beta}}^{p_0-1+\beta} + \| u(t) \|_{L^{2^*}}^{2\theta_0}) \leq 0 .
\]

(4.2)

Thanks to the interpolation inequality, we obtain

\[
\| u(t) \|_{L^p} \leq \| u(t) \|_{L^{p_0-1+\beta}}^{1-\gamma} \| u(t) \|_{L^{2^*}}^{\gamma} = (\| u(t) \|_{L^{p_0-1+\beta}}^{p_0-1+\beta})^{\frac{\gamma}{p_0-1+\beta}} (\| u(t) \|_{L^{2^*}}^{2\theta_0})^{\frac{1-\gamma}{2\theta_0}} \leq (\| u(t) \|_{L^{p_0-1+\beta}}^{p_0-1+\beta} + \| u(t) \|_{L^{2^*}}^{2\theta_0})^{\frac{\gamma}{p_0-1+\beta} + \frac{1-\gamma}{2\theta_0}} ,
\]

(4.3)

where \( \frac{1}{p} = \frac{\gamma}{p_0-1+\beta} + \frac{1-\gamma}{2\theta_0} \). Note that \( 2^* \theta_0 > p \) since \( m_1 + m_2 \geq \alpha_0 \). By (4.2) and (4.3), we obtain

\[
\frac{d}{dt} \| u(t) \|_L^p + C \| u(t) \|_L^{2\theta_0} \leq 0 ;
\]
with
\[ \lambda_0 = \frac{1}{1 + \frac{\gamma}{\theta_0} \left( \frac{1}{2} - \frac{1}{2p} \right)} \in (0, 1). \]

Thus, \( y(t) = \|u(t)\|_{L^p} \) satisfies
\[ y'(t) + Cy^{\lambda_0}(t) \leq 0. \tag{4.4} \]
This implies that there exists a finite time \( T_0 > 0 \) such that \( y(t) = 0 \) for \( t > T_0 \).
Thus, we obtain (1.7).

Finally, to estimate \( T_0 \), we solve directly (4.4) and obtain
\[ y^{1-\lambda_0}(t) + Ct \leq y^{1-\lambda_0}(0) = \|u_0\|_{L^p}^{1-\lambda_0} p. \]
Thus, (1.8) follows. This completes the proof. \( \square \)

Acknowledgements. This research was funded by the University of Economics, Ho Chi Minh City, Vietnam.

REFERENCES


NGUYEN ANH DAO
INSTITUTE OF APPLIED MATHEMATICS, UNIVERSITY OF ECONOMICS HO CHI MINH CITY, VIET NAM
Email address: anhdon@ueh.edu.vn