INFINITELY MANY SOLUTIONS FOR A SINGULAR SEMILINEAR PROBLEM ON EXTERIOR DOMAINS

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Abstract. In this article we prove the existence of an infinite number of radial solutions to \( \Delta U + K(x)f(U) = 0 \) on the exterior of the ball of radius \( R > 0 \) centered at the origin in \( \mathbb{R}^N \) with \( U = 0 \) on \( \partial B_R \), and \( \lim_{|x| \to \infty} U(x) = 0 \) where \( N > 2 \), \( f(U) \sim -\frac{1}{|U|^{q-1}U} \) for small \( U \neq 0 \) with \( 0 < q < 1 \), and \( f(U) \sim |U|^{p-1}U \) for large \( |U| \) with \( p > 1 \). Also, \( K(x) \sim |x|^{-\alpha} \) with \( \alpha > 2(N-1) \) for large \( |x| \).

1. Introduction

In this article we consider the problem

\[
\Delta U + K(|x|)f(U) = 0, \quad x \in \mathbb{R}^N \setminus B_R, \quad (1.1)
\]
\[
U = 0 \quad \text{on} \quad \partial (\mathbb{R}^N \setminus B_R), \quad (1.2)
\]
\[
U \to 0 \quad \text{as} \quad |x| \to \infty \quad (1.3)
\]

where \( U : \mathbb{R}^N \to \mathbb{R} \) with \( N > 2 \), \( B_R \) is the ball of radius \( R > 0 \) centered at the origin in \( \mathbb{R}^N \) and \( K(x) > 0 \).

We use the following assumptions:

(H1) \( f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \) and \( f \) odd, locally Lipschitz, and there exists \( \beta > 0 \) such that \( f < 0 \) on \( (0, \beta) \), \( f > 0 \) on \( (\beta, \infty) \).

(H2) \( f(U) = \frac{1}{|U|^{q-1}U} + g_1(U) \) for small \( U \neq 0 \), \( 0 < q < 1 \), \( g_1 \) is locally Lipschitz on \( \mathbb{R} \), \( g_1(0) = 0 \).

(H3) \( f(U) = |U|^{p-1}U + g_2(U) \) for large \( U \) where \( p > 1 \) and \( \lim_{U \to +\infty} \frac{g_2(U)}{|U|^p} = 0 \).

Now let \( F(U) = \int_0^U f(s) \, ds \). Since \( f \) is odd it follows that \( F \) is even, and from (H2) it follows that \( f \) is integrable near \( U = 0 \). Thus \( F \) is continuous and \( F(0) = 0 \). It also follows that \( F \) is bounded below by \( -F_0 \) with \( F_0 > 0 \).

(H4) there exists \( \gamma \) with \( 0 < \beta < \gamma \) such that \( F < 0 \) on \( (0, \gamma) \), \( F > 0 \) on \( (\gamma, \infty) \), \( F > -F_0 \) on \( \mathbb{R} \).

(H5) \( K \) and \( K' \) are continuous on \( [R, \infty) \) with \( K(r) > 0, 2(N-1) + \frac{rK'}{K} < 0 \), there exists \( \alpha \) such that \( \alpha > 2(N-1) \) and \( \lim_{r \to \infty} \frac{rK'}{K} = -\alpha \).

(H6) There exist \( K_1 > 0 \) and \( K_2 > 0 \) such that

\[
\frac{K_1}{r^\alpha} \leq K(r) \leq \frac{K_2}{r^\alpha} \quad \text{on} \quad [R, \infty).
\]

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Since we are interested in studying radial solutions of (1.1)–(1.3), we rewrite these equations with $r = |x|$, $U(r) = U(|x|)$ and see that $U$ satisfies:

$$U''(r) + \frac{N-1}{r}U'(r) + K(r)f(U(r)) = 0 \quad \text{on } (R, \infty), \quad (1.4)$$

$$U(R) = 0, \quad \lim_{r \to \infty} U(r) = 0. \quad (1.5)$$

Since $f(U)$ is discontinuous at $U = 0$ it follows that $U''$ is not continuous at any point where $U = 0$. However we will see that $U, U'$ are continuous on $[R, \infty)$ and satisfy

$$r^{N-1}U'(r) = \int_r^\infty s^{N-1}K(s)f(U(s)) \, ds. \quad (1.6)$$

In this article we prove the following result.

**Theorem 1.1.** Assuming (H1)–(H6) hold and $N > 2$, there exist an infinite number of nontrivial radial solutions of (1.5) and (1.6). In addition, for each nonnegative integer $n$, there is a solution of (1.5) and (1.6) with exactly $n$ zeros on $(0, R^{2-N})$.

The existence of a positive solution of (1.1) on $\mathbb{R}^N$ with $K(r) \equiv 1$ has been studied extensively [2, 3, 9] [12]. Recently the exterior domain $\mathbb{R}^N \setminus B_R(0)$ has been studied in [6, 7, 8, 10, 11, 13]. In addition, $f(U) = -|U|^{q-1}U + |U|^{p-1}U$ with $(1 < q < p)$ was studied in [11]. $f(U) = |U|^{q-1}U + g(U)$ with $(1 < q < p + 1)$ was studied in [1]. Also $f(U) = -|U|^{-q-1}U + g(U)$ with $(0 < q < 1)$ was studied in [12].

2. Preliminaries

We first prove the existence of a solution of (1.4) with

$$U(R) = 0 \quad \text{and} \quad U'(R) = a > 0 \quad (2.1)$$

on some neighborhood to the right of $R$. We denote this solution by $U_a(r)$ to emphasize the dependence on the initial parameter $a$. To prove existence of (1.4), (2.1) we make the change of variables

$$U_a(r) = V_a(r^{2-N}). \quad (2.2)$$

Then

$$U_a'(r) = (2-N)r^{1-N}V_a'(r^{2-N}),$$

$$U_a''(r) = (2-N)(1-N)r^{-N}V_a'(r^{2-N}) + (2-N)^2r^{2(1-N)}V_a''(r^{2-N}).$$

Letting $t = r^{2-N}$ and $r = t^{\frac{1}{N-2}}$ in (4), (7) we obtain

$$V_a''(t) + h(t)f(V_a(t)) = 0 \quad \text{on } (0, R^{2-N}), \quad (2.3)$$

$$V_a(R^{2-N}) = 0, \quad V_a'(R^{2-N}) = \frac{-aR^{N-1}}{N-2} < 0, \quad (2.4)$$

where from (H5) and (H6),

$$h(t) = \frac{1}{(N-2)^2}t^{\frac{2(N-1)}{N-2}}K(t^{\frac{1}{N-2}}) \sim \frac{t^{\tilde{\alpha}}}{(N-2)^2}, \quad \tilde{\alpha} = \frac{\alpha - 2(N-1)}{N-2} > 0 \quad (2.5)$$

on $(0, R^{2-N})$. Also from (H5) and (H6) it follows that there are constants $h_1, h_2$ with $0 < h_1 \leq h_2$ such that

$$h'(t) > 0, \quad h_1t^{\tilde{\alpha}} \leq h(t) \leq h_2t^{\tilde{\alpha}} \quad \text{on } (0, R^{2-N}). \quad (2.6)$$
For the existence of a solution of (2.3) on \((R^{2-N} - \epsilon, R^{2-N})\) with (2.4) for some \(\epsilon > 0\) we proceed as follows. First, integrate (2.3) on \((t, R^{2-N})\) and use (2.4). This gives

\[
-V_a(t) = \frac{aR^{N-1}}{N-2} - \int_t^{R^{2-N}} h(s)f(V_a(s)) \, ds. \tag{2.7}
\]

Integrating again over \((t, R^{2-N})\) and using (2.4) gives

\[
V_a(t) = \frac{aR^{N-1}}{N-2} (R^{2-N} - t) - \int_t^{R^{2-N}} \int_s^{R^{2-N}} h(x)f(V_a(x)) \, dx \, ds. \tag{2.8}
\]

Now let \(W(t) = \frac{V_a(t)}{R^{2-N} - t}\) so \(V_a(t) = (R^{2-N} - t)W(t)\) and

\[
W(R^{2-N}) \equiv \lim_{t \to (R^{2-N})^-} \frac{V_a(t)}{R^{2-N} - t} = -V'_a(R^{2-N}) = \frac{aR^{N-1}}{N-2}.
\]

Rewriting (2.8), we have

\[
W(t) = \frac{aR^{N-1}}{N-2} - \frac{1}{R^{2-N} - t} \int_t^{R^{2-N}} \int_s^{R^{2-N}} h(x)f((R^{2-N} - x)W(x)) \, dx \, ds. \tag{2.9}
\]

We now solve this equation on \([R^{2-N} - \epsilon, R^{2-N})\] by a fixed point method. Let \(a > 0, 0 < \epsilon < 1\), and let us define

\[
S = \left\{ W \in C[R^{2-N} - \epsilon, R^{2-N}]: W(R^{2-N}) = \frac{aR^{N-1}}{N-2}, \right. \\
\left. |W(t) - \frac{aR^{N-1}}{N-2}| \leq \frac{aR^{N-1}}{2(N-2)} \text{ on } [R^{2-N} - \epsilon, R^{2-N}] \right\}
\]

where \(C[R^{2-N} - \epsilon, R^{2-N}]\) is the set of real-valued continuous functions on \([R^{2-N} - \epsilon, R^{2-N}]\). Let

\[
||W|| = \sup_{x \in [R^{2-N} - \epsilon, R^{2-N}]} |W(x)|.
\]

Then \((S, || \cdot ||)\) is a Banach space. Now let us define a map \(T\) on \(S\) by \(TW(R^{2-N}) = \frac{aR^{N-1}}{N-2}\) and

\[
TW(t) = \frac{aR^{N-1}}{N-2} - \frac{1}{R^{2-N} - t} \int_t^{R^{2-N}} \int_s^{R^{2-N}} h(x)f((R^{2-N} - x)W(x)) \, dx \, ds \tag{2.10}
\]

on \((R^{2-N} - \epsilon, R^{2-N})\). Since \(W(x) \in S\) and \(0 < \epsilon < 1\) we have

\[
0 < \frac{aR^{N-1}}{2(N-2)} \leq W(x) \leq \frac{3aR^{N-1}}{2(N-2)} \text{ on } [R^{2-N} - \epsilon, R^{2-N}]. \tag{2.11}
\]

From (H2) we see \(g_1(x)\) is locally Lipschitz and \(g_1(0) = 0\) therefore it follows that

\[
|g_1((R^{2-N} - x)W(x))| \leq L|R^{2-N} - x||W(x)| \tag{2.12}
\]

where \(L\) is the Lipschitz constant for \(g_1\) on \([0, \frac{3aR^{N-1}}{2(N-2)}]\). It follows from (2.11) that

\[
\left| \frac{-1}{(R^{2-N} - x)^q}W(x) \right| \leq \frac{2^q(N-2)^q(R^{2-N} - x)^{-q}}{a^q(R^{N-1})^q} \tag{2.13}
\]
and using (2.6), (2.12), and (2.13) we see that

\[ |h(x)f((R^{2-N} - x)W(x))| \]

\[ = |h(x)\left(\frac{-1}{(R^{2-N} - x)qW(x)} + g_1((R^{2-N} - x)W(x))\right)| \]

\[ \leq h(R^{2-N}) \left[ \frac{2^q(N-2)^q(R^{2-N} - x)^{-q}}{a^q(R^{N-1})^q} + L[(R^{2-N} - x) 3aR^{N-1}] \right]. \]  

(2.14)

Integrating once we obtain

\[ \int_t^{R^{2-N}} |h(x)f((R^{2-N} - x)W(x))| \, dx \]

\[ \leq h(R^{2-N}) \left[ \frac{C_1}{a^q}(R^{2-N} - t)^{1-q} + C_2a(R^{2-N} - t)^2 \right] \]

where

\[ C_1 = \frac{2^q(N-2)^q}{(R^{N-1})^q(1-q)}, \quad C_2 = \frac{3LR^{N-1}}{4(N-2)}. \]

Thus from (2.15) we have

\[ \int_t^{R^{2-N}} |h(x)f((R^{2-N} - x)W(x))| \, dx \to 0 \quad \text{as } t \to (R^{2-N})^- \].  

(2.16)

Next integrating (2.15) on \((t, R^{2-N})\) and dividing by \((R^{2-N} - t)\) we obtain

\[ \frac{1}{R^{2-N} - t} \int_t^{R^{2-N}} \int_s^{R^{2-N}} |h(x)f((R^{2-N} - x)W(x))| \, dx \, ds \]

\[ \leq h(R^{2-N}) \left[ \frac{C_3(R^{2-N} - t)^{1-q}}{a^q} + aC_4(R^{2-N} - t)^2 \right] \]

(2.17)

where \(C_3 = \frac{C_1}{a^q}\) and \(C_4 = \frac{C_2}{a^q}\). Thus from (2.17) we see that

\[ \lim_{t \to (R^{2-N})^-} \frac{1}{R^{2-N} - t} \int_t^{R^{2-N}} \int_s^{R^{2-N}} |h(x)f((R^{2-N} - x)W(x))| \, dx \, ds = 0 \].  

(2.18)

Now we show that \(T : S \to S\) is a contraction mapping with \(T(W) \in S\) for each \(W \in S\) if \(\epsilon > 0\) is sufficiently small. First, let \(W \in S\) and so it follows from (2.17) and (2.18) that

\[ \frac{1}{R^{2-N} - t} \int_t^{R^{2-N}} \int_s^{R^{2-N}} h(x)f((R^{2-N} - x)W(x)) \, dx \, ds \]

is continuous on \([R^{2-N} - \epsilon, R^{2-N}]\). Then from (2.10), (2.17), and (2.18) we see that \(\lim_{t \to (R^{2-N})^-} TW(t) = aR^{N-1}/N-2\),

\[ |TW(t) - aR^{N-1}/N-2| \leq \frac{aR^{N-1}}{2(N-2)} \quad \text{on } [R^{2-N} - \epsilon, R^{2-N}] \]

and \(TW\) is continuous if \(\epsilon > 0\) is sufficiently small. Thus \(T : S \to S\) if \(\epsilon\) is sufficiently small. We next prove that \(T\) is a contraction mapping if \(\epsilon\) is sufficiently small. Let \(W_1, W_2 \in S\). Then

\[ TW_1(t) - TW_2(t) = -\frac{1}{R^{2-N} - t} \int_t^{R^{2-N}} \int_s^{R^{2-N}} h(x)[f((R^{2-N} - x)W_1(x)) - f((R^{2-N} - x)W_2(x))] \, dx \, ds. \]  

(2.19)
By (H2) we have \( f((R^2-N-x)W(x)) = -(R^2-N-x)^{-q}W^{-q}(x) + g_1((R^2-N-x)W(x)) \) where 0 < \( q < 1 \). Then by (2.12) and (2.13) we first estimate
\[
|f((R^2-N-x)W_1) - f((R^2-N-x)W_2)| = \left| \frac{-1}{(R^2-N-x)^q} \left[ \frac{1}{W_1^q} - \frac{1}{W_2^q} \right] + g_1((R^2-N-x)W_1) - g_1((R^2-N-x)W_2) \right| \tag{2.20}
\]
where \( L \) is again the Lipschitz constant for \( g_1 \) on \([0, \frac{3aR^{N-1}}{2(N-2)}] \). Next applying the mean value theorem we see that the right-hand side of (2.20) is equal to
\[
\frac{1}{(R^2-N-x)^q} \left[ \frac{q}{W_3^{q+1}} |W_1 - W_2| + L(R^2-N-x)|W_1 - W_2|, \right.
\]
where \( W_3 \) is between \( W_1 \) and \( W_2 \). Since \( W_i \in S \) for \( i = 1, 2, 3 \), and \( |W_i - \frac{aR^{N-1}}{2(N-2)}| \leq \frac{aR^{N-1}}{2(N-2)} \leq W_i \leq \frac{3aR^{N-1}}{2(N-2)} \) on \([R^2-N-\epsilon, R^2-N] \). Therefore \( W_3^{q+1} \geq \left( \frac{aR^{N-1}}{2(N-2)} \right)^{q+1} \), and so on \([R^2-N-\epsilon, R^2-N] \) we have
\[
|f((R^2-N-x)W_1) - f((R^2-N-x)W_2)| \leq |W_1 - W_2| \left[ \frac{q}{(R^2-N-x)^q} \left( \frac{2(N-2)}{aR^{N-1}} \right)^{q+1} + L(R^2-N-x) \right]. \tag{2.21}
\]
Recalling from (2.5) that \( h(t) \) is positive, continuous and increasing on \((0, R^2-N] \), with \( \alpha > 2(N-1) \) we see that
\[
|TW_1 - TW_2| \leq \frac{h(R^2-N)}{R^2-N - t} \int_t^{R^2-N} \int_s^{R^2-N} |W_1 - W_2| \left[ \frac{q}{(R^2-N-x)^q} \left( \frac{2(N-2)}{aR^{N-1}} \right)^{q+1} + L(R^2-N-x) \right] dx \, ds \leq \frac{h(R^2-N)}{R^2-N - t} \|W_1 - W_2\| \int_t^{R^2-N} \int_s^{R^2-N} \frac{q}{(R^2-N-x)^q} \left( \frac{2(N-2)}{aR^{N-1}} \right)^{q+1} + L(R^2-N-x) \right] dx \, ds \leq h(R^2-N) \|W_1 - W_2\| \left[ \frac{C_5 \epsilon^{1-q}}{a^{q+1}} + C_6 \epsilon^2 \right] = C_{7,\epsilon} \|W_1 - W_2\|.
\]
where
\[
C_5 = \frac{q}{(2-q)(1-q)} \left( \frac{2(N-2)}{aR^{N-1}} \right)^{q+1}, \quad C_6 = \frac{L}{6}, \quad C_{7,\epsilon} = h(R^2-N) \left[ \frac{C_5 \epsilon^{1-q}}{a^{q+1}} + C_6 \epsilon^2 \right].
\]
Since
\[
\lim_{\epsilon \to 0^+} C_{7,\epsilon} = \lim_{\epsilon \to 0^+} h(R^2-N) \left[ \frac{C_5 \epsilon^{1-q}}{a^{q+1}} + C_6 \epsilon^2 \right] = 0,
\]
for \( \epsilon \) sufficiently small we see that 0 < \( C_{7,\epsilon} < 1 \), and therefore it follows from (2.22) that \( T \) is a contraction. Then by the contraction mapping principle on \( S \) we see there exists a unique solution \( W \in S \) to \( TW = W \) on \([R^2-N-\epsilon, R^2-N] \) for some


$\epsilon > 0$. Then $V_\alpha(t) = (R^{2-N} - t)W(t)$ is a solution of (2.3) and satisfies (2.4) for some $\epsilon > 0$.

Now define the energy of solutions to (2.3) and (2.4) as

$$E_\alpha(t) = \frac{1}{2} V_\alpha''(t) + F(V_\alpha(t)).$$

(2.23)

Differentiating $E_\alpha$, using (2.3), and using that from (2.6) that $h'(t) > 0$, we have

$$E_\alpha'(t) = -\frac{V_\alpha'(t)h'(t)}{2h^2(t)} \leq 0.$$

(2.24)

Thus $E_\alpha$ is non-increasing where it is defined. Therefore for these $t$ with $t < R^{2-N}$ we have

$$0 < \frac{1}{2} \frac{a^2 R^{2(N-1)}}{(N-2)^2 h(R^{2-N})} = E_\alpha(R^{2-N}) \leq E_\alpha(t) = \frac{1}{2} \frac{V_\alpha'(t)}{h(t)} + F(V_\alpha(t)).$$

(2.25)

**Remark 2.1.** It follows from (2.3) that if $V_\alpha(t_0) \neq 0$ then $V_\alpha''(t_0)$ is defined and $V_\alpha''$ is continuous in a neighborhood of $t_0$. We also note if $V_\alpha$ is a solution of (2.7) and there exists a $Z_a \in (0, R^{2-N})$ such that $V_\alpha(Z_a) = 0$, then from (2.25) we see $0 < E_\alpha(Z_a) = \frac{1}{2} \frac{V_\alpha''(Z_a)}{h(Z_a)}$ and so $V_\alpha'(Z_a) \neq 0$. We also observe that if $V_\alpha(Z_0) = 0$ then it follows from (2.3) and (H2) that $V_\alpha''(Z_0)$ is undefined and that $\lim_{t \to Z_0^+} |V_\alpha''(t)| = \infty$. Therefore due to these considerations for the rest of this paper we will seek functions $V_\alpha$ that are continuously differentiable on $[0, R^{2-N}]$ and satisfy (2.7).

**Lemma 2.2.** Assume -(H1)-(H6) hold, $N > 2$, and $a > 0$. Let $V_\alpha(t)$ be the solution of (2.7) on $(R^{2-N} - \epsilon, R^{2-N})$ whose existence we have just proved. Then $V_\alpha$ and $V_\alpha'$ are defined and continuous on $[0, R^{2-N}]$. Also $|V_\alpha'(t)| \leq \frac{aR^{N-1}}{N-2} + 2F_0h(R^{2-N})$ on $[0, R^{2-N}]$, $|V_\alpha(t)| \leq \frac{aR^{N-1}}{N-2} + R^{2-N} \sqrt{2F_0h(R^{2-N})}$ on $[0, R^{2-N}]$, and $V_\alpha(t)$ satisfies (2.7) on $[0, R^{2-N}]$.

**Proof.** It follows from (2.3) that

$$\left(\frac{1}{2} V_\alpha'(t) + h(t)F(V_\alpha(t))\right)' = h'(t)F(V_\alpha(t)).$$

(2.26)

Integrating from $t$ to $R^{2-N}$ and using (2.4) yields

$$-\frac{1}{2} V_\alpha'(t) - h(t)F(V_\alpha(t)) = -\frac{1}{2} \frac{a^2 R^{2(N-1)}}{(N-2)^2} + \int_t^{R^{2-N}} h'(s)F(V_\alpha(s)) \, ds.$$

Since $-F_0 < F$ by (H4) and $h > 0, h' > 0$ by (2.6) then $hF_0 \geq -hF$ thus

$$-\frac{1}{2} V_\alpha'(t) + h(t)F_0 \geq -\frac{1}{2} \frac{a^2 R^{2(N-1)}}{(N-2)^2} + \int_t^{R^{2-N}} h'(s)F(V_\alpha(s)) \, ds \geq -\frac{1}{2} \frac{a^2 R^{2(N-1)}}{(N-2)^2} - F_0 \int_t^{R^{2-N}} h'(s) \, ds \geq -\frac{1}{2} \frac{a^2 R^{2(N-1)}}{(N-2)^2} - F_0 (h(R^{2-N}) - h(t)).$$
Therefore,

$$V_a^2(t) \leq \frac{a^2 R^{2(N-1)}}{(N-2)^2} + 2F_0 h(R^{2-N}).$$

Finally since $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$ for $x \geq 0$ and $y \geq 0$ we see that

$$|V_a'(t)| \leq \frac{aR^{N-1}}{N-2} + \sqrt{2F_0 h(R^{2-N})}.$$  \hfill (2.27) Integrating on $(t, R^{2-N})$ and using (2.27), (2.28) we obtain

$$|V_a(t)| = \left| \int_t^{R^{2-N}} V_a'(s) \, ds \right|$$

$$\leq \int_t^{R^{2-N}} |V_a'(s)| \, ds$$

$$\leq \int_t^{R^{2-N}} \left( \frac{aR^{N-1}}{N-2} + \sqrt{2F_0 h(R^{2-N})} \right) ds$$

$$= (R^{2-N} - t) \left( \frac{aR^{N-1}}{N-2} + \sqrt{2F_0 h(R^{2-N})} \right)$$

$$\leq \frac{aR}{N-2} + R^{2-N} \sqrt{2F_0 h(R^{2-N})}.$$  \hfill (2.28)

From (2.27) and (2.28) it follows that $V_a$ and $V_a'$ are bounded where they are defined and hence $V_a, V'_a$ exist on $[0, R^{2-N}]$ and $V_a'$ satisfies (2.7) on $[0, R^{2-N}]$. This completes the proof of Lemma 2.2. \hfill □

**Lemma 2.3.** Assume (H1)–(H6) hold, $N > 2$, $a > 0$, and $V_a(t)$ solves (13). Then the solutions $V_a(t)$ depend continuously on the parameter $a > 0$ on $[0, R^{2-N}]$.

**Proof.** First, let $0 < a_1 < a_2$. It follows from (2.27) and (2.28) that $V_a'$ and $V_a$ are bounded on $[0, R^{2-N}]$ and these upper bounds can be chosen to be independent of $a$ for $0 < a_1 \leq a \leq a_2$. Then from (2.27) and (2.28) we have

$$|V_a'(t)| \leq C_8 a_2 + C_9 \quad \forall a \text{ with } 0 < a_1 \leq a \leq a_2$$  \hfill (2.29)

where $C_8 = \frac{R^{2-N}}{N-2}$, $C_9 = \sqrt{2F_0 h(R^{2-N})}$, and

$$|V_a(t)| \leq C_{10} a_2 + C_{11} \quad \forall a \text{ with } 0 < a_1 \leq a \leq a_2$$  \hfill (2.30)

where $C_{10} = \frac{R}{N-2}$ and $C_{11} = R^{2-N} C_9$. Thus we see that $|V_a'|$ and $|V_a|$ are uniformly bounded on $[0, R^{2-N}]$ for all $a$ with $0 < a_1 \leq a \leq a_2$. Next, we suppose there exists $a^* > 0$ and we want to show that $V_a \to V_{a^*}$ uniformly on $[0, R^{2-N}]$ as $a \to a^*$. By way of contradiction suppose not. Then there exist $a_j$ such that $a_j \to a^*$ as $j \to \infty$, $t_j \in [0, R^{2-N}]$ and there is an $\epsilon_0 > 0$ such that

$$|V_{a_j}(t_j) - V_{a^*}(t_j)| \geq \epsilon_0 \quad \forall j.$$  \hfill (2.31)

Since $a_j \to a^*$ as $j \to \infty$ then if $j$ is sufficiently large we have $|a_j| \leq a^* + 1$ and by (2.29), (2.30) we see that $V_{a_j}$ and $V_{a^*}'$ are uniformly bounded and therefore equicontinuous on $[0, R^{2-N}]$. Then by the Arzela-Ascoli theorem there is a subsequence $a_{j_l}$, of $a_j$, such that $V_{a_{j_l}} \to V_{a^*}$ uniformly on $[0, R^{2-N}]$. So as $l \to \infty$,

$$0 \leq |V_{a_{j_l}}(t_{j_l}) - V_{a^*}(t_{j_l})| \geq \epsilon_0$$

which is impossible. Thus $V_a$ varies continuously with $a$ on $[0, R^{2-N}]$ for all $a$ with $0 < a_1 \leq a \leq a_2$. This completes the proof of Lemma 2.3. \hfill □
Lemma 2.4. Assume (H1)–(H6), \( N > 2 \), and let \( V_a(t) \) be the solution of (2.7). If \( a \) is sufficiently large then \( V_a(t) \) has a local maximum, \( M_a \), and a zero, \( Z_a \), with \( 0 < Z_a < M_a < R^{2-N} \). Further \( V_a(M_a) \rightarrow \infty, M_a \rightarrow R^{2-N}, Z_a \rightarrow R^{2-N}, \) and \( |V'_a(Z_a)| \rightarrow \infty \) as \( a \rightarrow \infty \).

Proof. We first show that if \( a \) is sufficiently large then there exists \( t_{a,\gamma} > 0 \) such that \( V_a(t_{a,\gamma}) = \gamma \) and \( 0 < V_a < \gamma \) on \( (t_{a,\gamma}, R^{2-N}) \). Suppose not. Then \( 0 < V_a(t) < \gamma \) on \( (0, R^{2-N}) \) and all sufficiently large \( a \). Since \( E_a \) is non-increasing on \( 0 < t < R^{2-N} \) and \( |V_a| < \gamma \) then \( F(V_a) < 0 \) and from (2.25) it follows that
\[
\frac{1}{2} V_a''(t) + F(V_a(t)) \geq 1 \frac{a^2 R^{2(N-1)}}{(N-2)h(R^{2-N})} > 0.
\]
Thus \( V_a' < 0 \) on \( (t, R^{2-N}) \) and we obtain
\[
-V_a'(t) \geq \frac{a R^{N-1}}{(N-2)\sqrt{h(R^{2-N})}} \sqrt{h(t)}.
\]
Integrating (2.33) from \( t \) to \( R^{2-N} \) gives
\[
V_a(t) = \int_t^{R^{2-N}} -V_a'(s) \, ds \geq \int_t^{R^{2-N}} \frac{a R^{N-1}}{(N-2)\sqrt{h(R^{2-N})}} \sqrt{h(s)} \, ds.
\]
Evaluating this expression at \( t = 0 \) we obtain
\[
\gamma \geq V_a(0) \geq \frac{a R^{N-1}}{(N-2)\sqrt{h(R^{2-N})}} \int_0^{R^{2-N}} \sqrt{h(s)} \, ds.
\]
The right-hand side approaches infinity as \( a \) goes to infinity which contradicts the assumption that the left-hand side is bounded by \( \gamma \). Thus \( V_a \) gets larger than \( \gamma \) as \( a \rightarrow \infty \) and so there exists \( t_{a,\gamma} \) with \( 0 < t_{a,\gamma} < R^{2-N} \) such that \( V_a(t_{a,\gamma}) = \gamma \) and \( 0 < V_a(t) < \gamma \) on \( (t_{a,\gamma}, R^{2-N}) \). In addition, evaluating (2.34) at \( t = t_{a,\gamma} \) we obtain
\[
\gamma = V_a(t_{a,\gamma}) \geq \frac{a R^{N-1}}{(N-2)\sqrt{h(R^{2-N})}} \int_{t_{a,\gamma}}^{R^{2-N}} \sqrt{h(s)} \, ds.
\]
Thus we see that
\[
t_{a,\gamma} \rightarrow R^{2-N} \quad \text{as} \quad a \rightarrow \infty.
\]
It then follows immediately that there is \( t_{a,\beta} \) such that \( t_{a,\gamma} < t_{a,\beta} < R^{2-N} \) and \( V_a(t_{a,\beta}) = \beta \). Since \( t_{a,\gamma} \rightarrow R^{2-N} \) as \( a \rightarrow \infty \) then it follows that
\[
t_{a,\beta} \rightarrow R^{2-N} \quad \text{as} \quad a \rightarrow \infty.
\]
Next we show that if \( V_a \) is decreasing for all \( t \in \left[\frac{1}{2} R^{2-N}, R^{2-N}\right] \) then we have \( \lim_{a \to \infty} V_a \left(\frac{1}{2} R^{2-N}\right) = \infty \). We suppose by the way of contradiction that \( V_a \left(\frac{1}{2} R^{2-N}\right) \leq A \) where \( A > 0 \) does not depend on \( a \) for a large. For \( \frac{1}{2} R^{2-N} \leq t \leq R^{2-N} \) it follows that there exists \( B > 0 \) such that \( F(V_a) < B \) on \( \left[\frac{1}{2} R^{2-N}, R^{2-N}\right] \) and all large \( a \). Since \( E_a \) is non-increasing,
\[
\frac{1}{2} V_a''(t) + B \geq \frac{1}{2} V_a''(t) + F(V_a(t)) = E_a(t) \geq E_a(R^{2-N}) = \frac{a^2 R^{2(N-1)}}{2(N-2)^2h(R^{2-N})}
\]
on \( \left[\frac{R^{2-N}}{2}, R^{2-N}\right] \). Rewriting the above expression we have
\[
-V_a'(t) \geq \sqrt{\frac{a^2 R^{2(N-1)}}{(N-2)^2h(R^{2-N})} - 2B \sqrt{h(t)}} \quad \text{on} \quad \left[\frac{R^{2-N}}{2}, R^{2-N}\right].
\]
Integrating this on \((t, R^{2-N})\) we obtain:
\[
V_a(t) \geq \left( \frac{a^2 R^{2(N-1)}}{(N-2)^2 h(R^{2-N})} - 2B \right) \int_t^{R^{2-N}} \sqrt{h(s)} \, ds. \quad (2.39)
\]

Now evaluating (2.39) at \(t = \frac{R^{2-N}}{2}\) we have
\[
A \geq V_a\left( \frac{R^{2-N}}{2} \right) \geq \left( \frac{a^2 R^{2(N-1)}}{(N-2)^2 h(R^{2-N})} - 2B \right) \int_{\frac{R^{2-N}}{2}}^{R^{2-N}} \sqrt{h(s)} \, ds. \quad (2.40)
\]

As \(a \to \infty\), the right-hand side approaches infinity, which is a contradiction since we were assuming \(A\) is finite. Thus
\[
\lim_{a \to \infty} V_a\left( \frac{1}{2} R^{2-N} \right) = \infty \quad \text{if} \quad V_a \text{ is decreasing on } \left[ \frac{R^{2-N}}{2}, R^{2-N} \right]. \quad (2.41)
\]

We next show that if \(V_a\) is decreasing on \([R^{2-N}, R^{2-N}]\) then \(V_a\left( \frac{3R^{2-N}}{4} \right) \to \infty\) as \(a \to \infty\). From (2.38) we know \(t_{a,\beta} \to R^{2-N}\) as \(a \to \infty\) so for a sufficiently large we have \(\frac{R^{2-N}}{2} \leq t_{a,\beta}\) and \(V_a(t) > \beta\) on \([R^{2-N}/2, t_{a,\beta}]\). From (2.3) and (H3) we see that \(V_a''(t) < 0\) on \([R^{2-N}/2, \beta]\) for sufficiently large \(a\). Thus \(V_a(t)\) is concave down here so we have for \(0 \leq \lambda \leq 1, \)
\[
V_a\left( \frac{\lambda R^{2-N}}{2} + (1-\lambda)t_{a,\beta} \right) \geq \lambda V_a\left( \frac{R^{2-N}}{2} \right) + (1-\lambda)V_a(t_{a,\beta})
\]
\[
= \lambda V_a\left( \frac{R^{2-N}}{2} \right) + (1-\lambda)\beta
\]
\[
\geq V_a\left( \frac{R^{2-N}}{2} \right).
\]

Now for \(t \in \left[ \frac{R^{2-N}}{2}, t_{a,\beta} \right]\) we can write \(t = \frac{\lambda R^{2-N}}{2} + (1-\lambda)t_{a,\beta}\), i.e.
\[
\lambda = \frac{t_{a,\beta} - t}{t_{a,\beta} - \frac{R^{2-N}}{2}}
\]
and thus \(0 \leq \lambda \leq 1\), and we obtain
\[
V_a(t) \geq \frac{t_{a,\beta} - t}{t_{a,\beta} - \frac{R^{2-N}}{2}} V_a\left( \frac{R^{2-N}}{2} \right) \quad \text{on } \left[ \frac{R^{2-N}}{2}, t_{a,\beta} \right]. \quad (2.42)
\]

Evaluating at \(t = \frac{3R^{2-N}}{4}\) gives
\[
V_a\left( \frac{3R^{2-N}}{4} \right) \geq \frac{t_{a,\beta} - \frac{3R^{2-N}}{4}}{t_{a,\beta} - \frac{R^{2-N}}{2}} V_a\left( \frac{R^{2-N}}{2} \right). \quad (2.43)
\]

From (2.38) we saw that \(t_{a,\beta} \to R^{2-N}\) as \(a \to \infty\) thus for sufficiently large \(a\) we have \(\frac{t_{a,\beta} - \frac{3R^{2-N}}{4}}{t_{a,\beta} - \frac{R^{2-N}}{2}} \geq \frac{1}{3}\) and therefore (50) along with (2.41) gives
\[
V_a\left( \frac{3R^{2-N}}{4} \right) \geq \frac{1}{3} V_a\left( \frac{R^{2-N}}{2} \right) \to \infty \quad \text{as} \quad a \to \infty. \quad (2.44)
\]

Now let us show that \(V_a(t)\) has a local maximum \(M_a\) on \([R^{2-N}/2, R^{2-N}]\) if \(a\) is sufficiently large. Suppose not. Then \(V_a(t)\) is decreasing on \([R^{2-N}/2, R^{2-N}]\).
Next let
\[
I_a = \min_{\left[\frac{1}{2} R^{2-N}, \frac{3}{4} R^{2-N}\right]} \frac{h(t)f(V_a(t))}{V_a(t)}.
\] (2.45)

Since \(h(t) > 0\) is bounded from below on \(\left[\frac{1}{2} R^{2-N}, \frac{3}{4} R^{2-N}\right]\) then there is an \(h_0 > 0\) such that \(h(t) > h_0\) on \(\left[\frac{1}{2} R^{2-N}, \frac{3}{4} R^{2-N}\right]\). Since we are assuming \(V_a\) is decreasing on \(\left[\frac{1}{2} R^{2-N}, \frac{3}{4} R^{2-N}\right]\) for all \(a > 0\) sufficiently large and since by (2.44) we have \(V_a \left(\frac{3}{4} R^{2-N}\right) \to \infty\) as \(a \to \infty\), it therefore follows that \(V_a \to \infty\) uniformly on \(\left[\frac{1}{2} R^{2-N}, \frac{3}{4} R^{2-N}\right]\). By (H3) it then follows for sufficiently large \(a\) that \(\frac{f(V_a)}{V_a} \geq \frac{1}{2} V_a^{p-1}\) and therefore
\[
I_a = \min_{\left[\frac{1}{2} R^{2-N}, \frac{3}{4} R^{2-N}\right]} \frac{h(t)f(V_a)}{V_a} \\
\geq h_0 \min_{\left[\frac{1}{2} R^{2-N}, \frac{3}{4} R^{2-N}\right]} \frac{f(V_a)}{V_a} \\
\geq h_0 \left(\frac{3}{4} R^{2-N}\right)^{p-1} V_a^{p-1} \\
\geq h_0 \left(\frac{3}{4} R^{2-N}\right)^{p-1} \left(\frac{3}{4} R^{2-N}\right).
\]

By (2.44) the right-hand side goes to infinity, and thus we obtain
\[
\lim_{a \to \infty} I_a = \infty.
\] (2.46)

Now we apply the Sturm Comparison theorem [5] on \(\left[\frac{1}{2} R^{2-N}, \frac{3}{4} R^{2-N}\right]\). Consider
\[
V_a'' + \left(\frac{h(t)f(V_a)}{V_a}\right)V_a = 0,
\]
\[
W_a'' + I_a W_a = 0
\] (2.47) (2.48)

where
\[
\beta < V_a \left(\frac{3}{4} R^{2-N}\right) = W_a \left(\frac{3}{4} R^{2-N}\right),
\]
\[
V_a' \left(\frac{3}{4} R^{2-N}\right) = W_a' \left(\frac{3}{4} R^{2-N}\right) < 0.
\] (2.49) (2.50)

Since \(W_a'' + I_a W_a = 0\) and \(W_a \neq 0\), it follows that \(W_a = C_{12} \sin(\sqrt{\mathcal{M}_a} t) + C_{13} \cos(\sqrt{\mathcal{M}_a} t)\) where \(C_{12}\) and \(C_{13}\) are not both zero. It is well-known that any interval of length \(\frac{\pi}{\sqrt{\mathcal{M}_a}}\) has a zero of \(W_a\) and so it follows that \(W_a\) has a local maximum \(\mathcal{M}_a \in \left[\frac{1}{4} R^{2-N} - \frac{\pi}{\sqrt{\mathcal{M}_a}}, \frac{1}{4} R^{2-N}\right]\) and \(W_a\) is decreasing on \(\left[\mathcal{M}_a, \frac{1}{4} R^{2-N}\right]\). Also for \(a\) sufficiently large then from (2.47), \(\frac{3}{4} R^{2-N} - \frac{\pi}{\sqrt{\mathcal{M}_a}} > \frac{1}{2} R^{2-N}\). Multiplying (2.47) by \(W_a\), (2.48) by \(V_a\), and subtracting we obtain
\[
(W_a V_a' - V_a W_a')' + \left(\frac{h(t)f(V_a)}{V_a} - I_a\right)V_a W_a = 0.
\] (2.51)

Using (2.49), (2.50) and since \(W_a\) has a local maximum \(\mathcal{M}_a\) then integrating (2.51) on \([\mathcal{M}_a, \frac{1}{4} R^{2-N}]\) we obtain
\[
- W_a(\mathcal{M}_a) V_a'(\mathcal{M}_a) + \int_{\mathcal{M}_a}^{\frac{3}{4} R^{2-N}} \left(\frac{h(t)f(V_a)}{V_a} - I_a\right)V_a W_a = 0.
\] (2.52)
Since \( W_a(M_a) > W_a(\frac{1}{2}R^{2-N}) > \lambda > 0 \) by (2.49) and \( \left( \frac{h(t)f(V_a)}{V_a} - I_a \right)V_aW_a > 0 \) on \([M_a, \frac{3}{4}R^{2-N}]\) then \( \frac{1}{2}R^{2-N} \left( \frac{h(t)f(V_a)}{V_a} - I_a \right)V_aW_a > 0 \) and so it follows that \( V'_a(M_a) > 0 \) which is a contradiction to the assumption that \( V'_a(t) < 0 \) on \( [R^{2-N}, R^{2-N}] \). Thus \( V_a(t) \) must have a local maximum, \( M_a \), with \( \frac{1}{2}R^{2-N} < M_a < R^{2-N} \) and \( V_a \) decreasing on \((M_a, R^{2-N})\) if \( a \) is sufficiently large.

Now let us show that \( V_a(M_a) \to \infty \) as \( a \to \infty \). Suppose by the way of the contradiction that there exists a constant \( C_{14} > 0 \) independent of \( a \) such that \( V_a(M_a) < C_{14} \) and so \( V_a(t) < C_{14} \) on \((M_a, R^{2-N})\). Integrating (2.3) on \((M_a, R^{2-N})\) and using (2.4) gives

\[
\int_{M_a}^{R^{2-N}} V''_a(t) \, dt + \int_{M_a}^{R^{2-N}} h(t)f(V_a(t)) \, dt = 0.
\]

Therefore

\[
aR^{2-N} = \int_{M_a}^{R^{2-N}} h(t)f(V_a(t)) \, dt = \int_{M_a}^{R^{2-N}} h(t)(-V_a^{-q}(t)) \, dt + \int_{M_a}^{R^{2-N}} h(t)g_1(V_a(t)) \, dt \leq \int_{M_a}^{R^{2-N}} h(t)g_1(V_a(t)) \, dt.
\]

Since \( 0 \leq V_a(t) \leq V_a(M_a) \leq C_{14} \) and \( g_1(V) \leq C_{15} \) is continuous, \( g_1(V_a) \leq C_{15} \) for some constant \( C_{15} > 0 \) on \([M_a, R^{2-N}]\), and since \( h(t) \leq h_2t^\alpha \) (by (2.4)), estimating (2.53) gives

\[
aR^{2-N} \leq \frac{h_2C_{15}((R^{2-N})^{1+\alpha} - M_a^{1+\alpha})}{1+\alpha} \leq \frac{h_2C_{15}((R^{2-N})^{1+\alpha})}{1+\alpha}.
\]

The left-hand side of (2.54) goes to \(+\infty\) as \( a \to \infty \) but the right-hand side is bounded which contradicts the assumption that \( 0 \leq V_a(M_a) \leq C_{14} \). Thus

\[
V_a(M_a) \to \infty \text{ as } a \to \infty.
\]

Now let us show that \( \lim_{a \to \infty} M_a = R^{2-N} \). Since \( V''_a(t) \leq 0 \) on \((M_a, t_{a,\beta})\) then \( V_a \) is concave down here and so we obtain

\[
V_a(\lambda M_a + (1 - \lambda)t_{a,\beta}) \geq \lambda V_a(M_a) + (1 - \lambda)\beta \tag{2.56}
\]

where \( 0 \leq \lambda \leq 1 \). Letting \( \lambda = 1/2 \) gives

\[
V_a\left(\frac{M_a + t_{a,\beta}}{2}\right) \geq \frac{1}{2}V_a(M_a) + \frac{1}{2}\beta = \frac{V_a(M_a) + \beta}{2}. \tag{2.57}
\]

From (2.55) we know that \( V_a(M_a) \to \infty \) as \( a \to \infty \) so then (2.57) implies

\[
V_a\left(\frac{M_a + t_{a,\beta}}{2}\right) \to \infty \text{ as } a \to \infty. \tag{2.58}
\]

Since \( V_a \) is decreasing on \([M_a, \frac{M_a + t_{a,\beta}}{2}]\) it follows that \( V_a \to \infty \) uniformly on \([M_a, \frac{M_a + t_{a,\beta}}{2}]\) for sufficiently large \( a \). Since \( f(V_a(t)) \geq \frac{1}{2}V_a''(t) \) for \( V_a \) large by (H3), from (2.3) \(-V''_a(t) \geq f(V_a(t)) \geq \frac{1}{2}h(t)V_a''(t) \) on \([M_a, \frac{M_a + t_{a,\beta}}{2}]\). Since \( V_a \) is decreasing on \((M_a, t)\), integrating from \( M_a \) to \( t \) where \( M_a \leq t \leq \frac{M_a + t_{a,\beta}}{2} \) we obtain

\[
-V'_a(t) = -V'_a(t) + V'_a(M_a)
\]
Next we show there is a $Z_{t_{a,\beta}}$ so $E_{a} = 0$ on $(0, M_{a})$, and $Z_{a} \to R^{2-N}$ as $a \to \infty$. Moreover $V_{a}^{\prime}(Z_{a}) \to -\infty$ as $a \to \infty$. Again we do this by contradiction. Let us assume $V_{a}^{\prime}(t) > 0$ on $(0, M_{a})$. Since $E_{a}(t)$ is non-increasing then we have

$$M_{a} \to R^{2-N} \quad \text{as} \quad a \to \infty. \quad (2.62)$$

Next we show there is a $Z_{a} \in (0, M_{a})$ such that $V_{a}^{\prime}(Z_{a}) = 0$, $V_{a}(t) > 0$ on $(Z_{a}, R^{2-N})$, and $Z_{a} \to R^{2-N}$ as $a \to \infty$. Moreover $V_{a}^{\prime}(Z_{a}) \to -\infty$ as $a \to \infty$. Again we do this by contradiction. Let us assume $V_{a}(t) > 0$ on $(0, M_{a})$. Since $E_{a}(t)$ is non-increasing then we have

$$F(V_{a}(M_{a})) \leq \frac{1}{2} V_{a}^{2} + F(V_{a}(t)) \quad \text{for} \quad 0 \leq t \leq M_{a}. \quad (2.63)$$

Now if $V_{a}$ has a positive local minimum $m_{a}$, then $V_{a}^{\prime}(m_{a}) \geq 0$ so $f(V_{a}(m_{a})) \leq 0$ so $0 < V_{a}(m_{a}) \leq \beta$ but also $0 < E_{a}(m_{a}) = F(V_{a}(m_{a}))$ so $V_{a}(m_{a}) > \gamma \geq \beta$ which is a contradiction. Thus $V_{a}^{\prime} > 0$ on $(0, M_{a})$. Rewriting, integrating (2.63) over $[M_{a}, M_{a}^{+}]$, using (2.5), and making a change of variables gives

$$\int_{0}^{V_{a}(M_{a})} \frac{ds}{\sqrt{F(V_{a}(M_{a})) - F(s)}} \geq \int_{0}^{M_{a}} \frac{ds}{\sqrt{F(V_{a}(M_{a})) - F(s)}} \geq \int_{0}^{M_{a}} \frac{|V_{a}^{\prime}(t)| dt}{\sqrt{F(V_{a}(M_{a})) - F(V_{a}(t))}} \geq \int_{0}^{M_{a}} \frac{\sqrt{2h(s)} ds}{\sqrt{V_{a}^{2}(1 - \frac{1}{2+\alpha})}}$$

$$= \frac{\sqrt{2h} (1 - \frac{1}{2+\alpha})}{1 + \frac{\alpha}{2}} M_{a}^{1+\frac{\alpha}{2}}. \quad (2.64)$$
Now we estimate the left-hand side. It follows from (H3) that $f(U) \geq \frac{1}{2} U^p$ for $U$ sufficiently large therefore for $U$ large enough we see that $\min_{\frac{1}{2}U} f \geq \frac{1}{2U} U^p$ and since $p > 1$, it follows that

$$\lim_{U \to \infty} \frac{U}{\min_{\frac{1}{2}U, U}} f = 0. \quad (2.65)$$

We now estimate the integral on the left-hand side of (2.64) when $s \in [0, \frac{V_a(M_a)}{2}]$ and $a$ is sufficiently large. We then have $F(s) < F(\frac{V_a(M_a)}{2})$ for all $s \in (0, \frac{V_a(M_a)}{2})$ and thus $F(V_a(M_a)) - F(\frac{V_a(M_a)}{2}) < F(V_a(M_a)) - F(s)$ so

$$\int_{0}^{\frac{V_a(M_a)}{2}} \frac{ds}{\sqrt{F(V_a(M_a)) - F(s)}} \leq \int_{0}^{\frac{V_a(M_a)}{2}} \frac{ds}{\sqrt{F(V_a(M_a)) - F(\frac{V_a(M_a)}{2})}} \quad (2.66)$$

By the mean value theorem there is a $d_1 > 0$ such that $\frac{V_a(M_a)}{2} < d_1 < V_a(M_a)$ and

$$F(V_a(M_a)) - F(\frac{V_a(M_a)}{2}) = f(d_1)[V_a(M_a) - \frac{V_a(M_a)}{2}]$$

$$= f(d_1)[\frac{V_a(M_a)}{2}]$$

$$\geq \left[ \min_{\frac{V_a(M_a)}{2}, V_a(M_a)} f \right] \frac{V_a(M_a)}{2}$$

so

$$\frac{V_a(M_a)}{\sqrt{F(V_a(M_a)) - F(\frac{V_a(M_a)}{2})}} \leq \frac{\sqrt{\min_{\frac{V_a(M_a)}{2}, V_a(M_a)} f}}{\sqrt{\min_{\frac{V_a(M_a)}{2}, V_a(M_a)} f}} \to 0 \quad (2.67)$$

as $a \to \infty$, by (2.65). Thus by (2.66) and (2.67) we see that

$$\lim_{a \to \infty} \int_{0}^{\frac{V_a(M_a)}{2}} \frac{ds}{\sqrt{2\sqrt{F(V_a(M_a)) - F(s)}}} = 0. \quad (2.68)$$

Next, we estimate the integral on the left-hand side of (2.64) for $s \in \left[ \frac{V_a(M_a)}{2}, V_a(M_a) \right]$. By the mean value theorem there is a $d_2 > 0$ with $\frac{V_a(M_a)}{2} < d_2 < V_a(M_a)$ such that

$$F(V_a(M_a)) - F(s) = f(d_2)[V_a(M_a) - s] \geq \left[ \min_{\frac{V_a(M_a)}{2}, V_a(M_a)} f \right] [V_a(M_a) - s].$$
Therefore,
\[
\int_{V_a(M_a)}^{V_a(M_a)} \frac{ds}{\sqrt{F(V_a(M_a)) - F(s)}} \\
\leq \int_{V_a(M_a)}^{V_a(M_a)} \frac{ds}{\sqrt{\min_{|V_a(M_a)|} F'[V_a(M_a) - s]}} \\
= \sqrt{2} \sqrt{\min_{|V_a(M_a)|} \frac{V_a(M_a)}{F'[V_a(M_a)]}}.
\]

Thus by (2.65) we see that
\[
\lim_{a \to \infty} \int_{V_a(M_a)}^{V_a(M_a)} \frac{dt}{\sqrt{2 \sqrt{F(V_a(M_a)) - F(s)}}} = 0. \tag{2.70}
\]

Combining (2.67) and (2.70) we have
\[
\lim_{a \to \infty} \int_{0}^{V_a(M_a)} \frac{ds}{\sqrt{2 \sqrt{F(V_a(M_a)) - F(s)}}} = 0. \tag{2.71}
\]

Thus the left-hand side of (2.64) goes to 0 as \( a \to \infty \) but the right-hand side of (2.64) does not because by (2.62) we know \( M_a \to R^{2-N} \) as \( a \to \infty \) and so we get a contradiction. Thus for \( a \) sufficiently large \( V_a(t) \) has a first zero, \( Z_a \), with \( V_a(Z_a) = 0 \) and \( V_a(t) > 0 \) on \((Z_a, R^{2-N})\). Similarly rewriting (2.63) and integrating on \((Z_a, M_a)\) we obtain
\[
\int_{0}^{V_a(M_a)} \frac{ds}{\sqrt{2 \sqrt{F(V_a(M_a)) - F(s)}}} \geq \sqrt{h_1} \left( M_a^{1+\frac{\alpha}{2}} - Z_a^{1+\frac{\alpha}{2}} \right). \tag{2.72}
\]

Since the left-hand side approaches 0 as \( a \to \infty \) (by (2.71)), we see \( M_a^{1+\frac{\alpha}{2}} - Z_a^{1+\frac{\alpha}{2}} \to 0 \) as \( a \to \infty \). Also since we know from (2.62) that \( M_a \to R^{2-N} \) as \( a \to \infty \) this then implies that \( Z_a \to R^{2-N} \) as \( a \to \infty \).

Finally we show that \( V_a'(Z_a) \to +\infty \) as \( a \to \infty \). Since \( Z_a \to R^{2-N} \) as \( a \to \infty \) and \( E_a(t) \) is non-increasing, since \( 0 < Z_a \leq M_a \) we have
\[
0 < F(V_a(M_a)) = E_a(M_a) \leq E_a(Z_a) = \frac{1}{2} \frac{V_a'^2(Z_a)}{h(Z_a)}.
\]

and so rewriting this inequality gives
\[
2h(Z_a)F(V_a(M_a)) \leq V_a'^2(Z_a). \tag{2.73}
\]

As \( a \to \infty \) the left-hand side approaches \( \infty \) because \( \lim_{a \to \infty} h(Z_a) = h(R^{2-N}) > 0 \) and \( \lim_{a \to \infty} F(V_a(M_a)) = \infty \) by (2.55). Thus \( V_a'^2(Z_a) \to \infty \) as \( a \to \infty \) and thus it follows that \( V_a'(Z_a) \to +\infty \) as \( a \to \infty \). In similar way if \( a > 0 \) is sufficiently large then \( V_a(t) \) has a second zero \( Z_{a,2} \) on \((0, R^{2-N})\) with \( Z_{a,2} \to R^{2-N} \) as \( a \to \infty \) and \( V_a'(Z_{a,2}) \to -\infty \). More generally \( V_a(t) \) has \( n \) zeros on \((0, R^{2-N})\) if \( a > 0 \) is sufficiently large. This completes the proof.

\[\square\]

**Lemma 2.5.** Let \( V_a(t) \) be the solution of (2.7), (H1)–(H6) hold, and \( N > 2 \). If \( R \) is sufficiently large then \( V_a(t) > 0 \) for all \( t \in (0, R^{2-N}) \) if \( a \) sufficiently small.
Proof. To reach a contradiction, suppose there is $Z_n \in (0, R^{2-N})$ such that $V_a(Z_n) = 0$ for all $a$ sufficiently small. Then there exists $0 < M_a < R^{2-N}$ such that $V_a'(M_a) = 0$ and $V_a'(t) < 0$ on $(M_a, R^{2-N})$. Also $0 < E_a(M_a) = F(V_a(M_a))$ so $V_a(M_a) > \gamma$. Then by Lemma 2.2 we see that $|V_a''(t)| \leq \frac{aR^{2-N}}{N-2} + \sqrt{2F_0h(R^{2-N})}$, and since $V_a(t)$ is decreasing on $(M_a, R^{2-N})$ this gives

$$-V_a'(t) \leq \frac{aR^{2-N}}{N-2} + \sqrt{2F_0h(R^{2-N})} \text{ on } (M_a, R^{2-N}).$$

(2.74)

Integrating from $t$ to $R^{2-N}$ and using (2.4) we obtain:

$$V_a(t) \leq \left(\frac{aR^{2-N}}{N-2} + \sqrt{2F_0h(R^{2-N})}\right)(R^{2-N} - t) \leq \left(\frac{aR^{2-N}}{N-2} + \sqrt{2F_0h(R^{2-N})}\right)R^{2-N}.$$

Substituting $t = M_a$ gives

$$\gamma \leq \left(\frac{aR^{2-N}}{N-2} + \sqrt{2F_0h(R^{2-N})}\right)R^{2-N}.$$

Taking the limit as $a \to 0^+$ we obtain

$$\gamma \leq \sqrt{2F_0h(R^{2-N})}R^{2-N} - \sqrt{2F_0h_2(R^{2-N})^{\alpha/2}}R^{2-N}. \quad (2.75)$$

Then using (2.6) we obtain

$$\gamma \leq \sqrt{2F_0h_2R^{1-\frac{\alpha}{2}}} \quad \text{where } \alpha > 2(N-1). \quad (2.76)$$

Thus we see that the right-hand side of (2.76) is larger than $\gamma$ for $R$ sufficiently large but since $\alpha > 2$ we see the right-hand side goes to $0$ as $R \to \infty$ contradicting (2.76). Thus if $R$ is sufficiently large then $0 < V_a(t) < \gamma$ if $a$ is sufficiently small. This completes the proof. \hfill \Box

3. PROOF OF THE MAIN THEOREM [1.1]

Lemma 3.1. Assume $N > 2$ and (H1)–(H6) hold. For $a > 0$ let $V_a(t)$ be the solution of (2.7). Then $V_a(t)$ has at most a finite numbers of zeros on $(0, R^{2-N})$.

Proof. Suppose by way of contradiction that there are distinct zero’s $Z_n \in (0, R^{2-N})$ such that $V_a(Z_n) = 0$. Then either there is a decreasing subsequence (still labeled $Z_n$) or an increasing subsequence and a $Z^* \in [0, R^{2-N})$ such that $Z_n \to Z^*$ as $n \to \infty$. By continuity $V_a(Z^*) = 0$. Also since $V_a'(R^{2-N}) < 0$ there exists $\epsilon > 0$ such that $V_a$ is not zero on $(R^{2-N} - \epsilon, R^{2-N})$ and thus $Z^* \neq R^{2-N}$. Therefore $0 \leq Z^* < R^{2-N}$. Without loss of generality assume $Z_n$ is decreasing. Then there is a local maximum or local minimum $M_n$ of $V_a$ with $Z_{n+1} < M_n < Z_n$ so $M_n \to Z^*$ as $n \to \infty$ and notice also that since $E_a(t) > 0$ on $[0, R^{2-N}]$ by (2.25) then $E_a(M_n) = F(V_a(M_n)) > 0$ which implies that $|V_a(M_n)| > \gamma$. Now by the mean value theorem,

$$\gamma \leq |V_a(M_n)| = |V_a(M_n) - V_a(Z_n)| = |V_a'(c_n)||M_n - Z_n|,$$

(3.1)

where $c_n \neq 0$ and $M_n < c_n < Z_n$. Since $M_n \to Z^*$ and $Z_n \to Z^*$ it follows that $|M_n - Z_n| \to 0$ as $a \to \infty$. Also by (2.27) we see $|V_a'(c_n)| < \frac{aR^{2-N}}{N-2} + \sqrt{2F_0h(R^{2-N})} \leq \infty$. This implies that the right-hand side of (84) goes to zero which contradicts the fact that $\gamma > 0$. Thus $V_a$ has at most a finite numbers of zeros on $(0, R^{2-N})$. This completes the proof. \hfill \Box
Let 
\[ S_n = \{a > 0 : V_a(t) \text{ has exactly } n \text{ zeros on } (0, R^{2-N}) \} . \]

By Lemma 3.1 we know that \( S_n \) is nonempty for some \( n \). Let \( n_0 \geq 0 \) be the smallest non-negative integer \( n \) such that \( S_n \neq \emptyset \) (so \( S_{n_0} \neq \emptyset \) and \( S_0, S_1, S_2, \ldots, S_{n_0-1} \) are all empty). By Lemma 2.3 it follows that \( S_{n_0} \) is bounded above. Therefore the supremum of \( S_{n_0} \) exists, and so we let 
\[ a_{n_0} = \sup S_{n_0} . \]

If in addition \( R \) is sufficiently small then \( S_0 \neq \emptyset \) by Lemma 2.4 and so \( n_0 = 0 \).

**Lemma 3.2.** \( V_{a_n}(t) \) has exactly \( n \) zeros on \((0, R^{2-N})\) and \( V_{a_n}(0) = 0 \) for all \( n \geq n_0 \).

**Proof.** Since \( S_{n_0} \) is the smallest value of \( n \) such that \( S_n \neq \emptyset \) this implies that \( V_{a_{n_0}}(t) \) has at least \( n_0 \) zeros on \((0, R^{2-N})\). Next we show that \( V_{a_{n_0}}(t) \) has at most \( n_0 \) zeros on \((0, R^{2-N})\). By way of contradiction, suppose there exists an \((n_0+1)\)st zero \( Z^* \) with \( Z^* \in (0, R^{2-N}) \) such that \( V_{a_{n_0}}(Z^*) = 0 \) and \( 0 < Z^* < Z_{n_0} < \cdots < Z_1 < R^{2-N} \) and suppose without loss of generality that \( V_{a_{n_0}} > 0 \) on \((0, Z^*)\). Since \( E_a \) is non-increasing then \( 0 < E_a(Z^*) = \frac{1}{2} \frac{V_{a_{n_0}}^2(Z^*)}{h(Z^*)} \) which implies that \( V_{a_{n_0}}^2(Z^*) > 0 \). Since \( V_{a_{n_0}}' > 0 \) on \((0, Z^*)\) it follows that \( V_{a_{n_0}}(Z^*) < 0 \). So \( V_{a_{n_0}}(Z^* - \delta) > 0 \) for \( \delta > 0 \) sufficiently small. By continuity with respect to \( a \) it follows that if \( a < a_{n_0} \) then \( V_a \) also has a \((n_0+1)\)st zero on \((0, R^{2-N})\) which is a contradiction to the definition of \( a_{n_0} \). Therefore we see that \( V_{a_{n_0}}(t) \) has exactly \( n_0 \) zeros on \((0, R^{2-N})\). Now we denote \( Z_{a_{n_0}} \) as the \( n_0 \)th zero of \( V_{a_{n_0}}(t) \). Then \( V_{a_{n_0}}(t) \neq 0 \) if \( 0 < t < Z_{a_{n_0}} \).

So without loss of generality we assume that \( V_{a_{n_0}} < 0 \) on \((0, Z_{a_{n_0}})\). It follows by continuity of \( V_{a_{n_0}} \) that \( V_{a_{n_0}}(0) = \lim_{t \rightarrow 0^+} V_{a_{n_0}}(t) \leq 0 \). Thus \( V_{a_{n_0}}(0) \leq 0 \). Next we show that \( V_{a_{n_0}}(0) = 0 \). So suppose not. Then \( V_{a_{n_0}} < 0 \) on \((0, Z_{a_{n_0}})\).

From the remark before Lemma 2.2 we saw that \( V_{a_{n_0}}'(Z) \neq 0 \) if \( V_{a_{n_0}}(Z) = 0 \). For \( a_{n_0+1} > a > a_{n_0} \) we see that

\[ |V_a'| \leq |a_{n_0+1}| |\frac{R^{N-1}}{N-2} + \sqrt{2F_0 h(R^{2-N})}| \] (by Lemma 2.2)

It follows then that \( V_a \) will also have \( n_0 \) zeros on \((0, R^{2-N})\) if \( a_{n_0+1} > a > a_{n_0} \). On the other hand, if \( a > a_{n_0} \) then by the definition of \( a_{n_0} \) we see that \( V_a \) has at least \((n_0+1)\) zeros on \((0, R^{2-N})\) which is a contradiction. Thus the assumption that \( V_{a_{n_0}}(0) < 0 \) is false and since \( V_{a_{n_0}}(0) \leq 0 \) then it follows that \( V_{a_{n_0}}(0) = 0 \).

Next let 
\[ S_{n_0+1} = \{a > 0 : V_a(t) \text{ has exactly } n_0 + 1 \text{ zeros on } (0, R^{2-N}) \} . \]

For \( a \) slightly larger than \( a_{n_0} \) than \( V_a \) has at least \( n_0 + 1 \) zeros on \((0, R^{2-N})\) by definition of \( a_{n_0} \). Next we show that \( V_a(t) \) has at most \( n_0 + 1 \) zeros on \((0, R^{2-N})\) if \( a \) is close to \( a_{n_0} \) and \( a > a_{n_0} \). So suppose not and suppose that \( V_a \) has an \((n_0+2)\)nd zero on \((0, R^{2-N})\). Then \( V_a \) has a local maximum or a local minimum at some \( M_a \) where \( 0 < Z_{a_{n_0+2}} < M_a < Z_{a_{n_0+1}} \) and for \( a \) slightly larger than \( a_{n_0} \). Also \( \lim_{t \rightarrow a_{n_0}} V_a = V_{a_{n_0}} \) uniformly on \((0, R^{2-N})\) and \( Z_{a_{n_0+1}} \rightarrow 0 \), hence \( M_a \rightarrow 0 \) as \( a \rightarrow a_{n_0} \). Since \( 0 < E_a(M_a) = F(V_a(M_a)) \) it follows that \( |V_a(M_a)| > \gamma > \beta \) so \( \beta \leq |V_a(M_a)| \rightarrow |V_{a_{n_0}}(0)| = 0 \) which is false. Thus if \( a > a_{n_0} \) and \( a \) is close to \( a_{n_0} \) then \( V_a \) has at most \( n_0 + 1 \) zeros on \((0, R^{2-N})\) and since we showed earlier \( V_a \) has at least \( n_0 + 1 \) zeros on \((0, R^{2-N})\) then it follows that \( S_{n_0+1} \neq \emptyset \). By Lemma 2.2 it follows that \( S_{n_0+1} \) is bounded from above.
Let 
\[ a_{n_0+1} = \sup S_{n_0+1}. \]
In a similar fashion way we can show that \( V_{a_{n_0+1}}(t) \) has exactly \( n_0 + 1 \) zeros on \((0, R^{2-N})\) and \( V_{a_{n_0+1}}(0) = 0 \). Proceeding inductively we can show that for each \( n \in \mathbb{N} \) there exists a solution \( V_{a_{n_0+n}}(t) \) of (2.7) which has exactly \( n_0 + n \) zeros on \((0, R^{2-N})\) and \( V_{a_{n_0+n}}(0) = 0 \). This completes the proof of Lemma 3.2 and the proof of the main theorem. □

References


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