STANDING WAVES TO CHERN-SIMONS-SCHRÖDINGER SYSTEMS WITH CRITICAL EXPONENTIAL GROWTH

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Abstract. In this article we study the existence of standing waves to nonlinear Chern-Simons-Schrödinger systems with critical exponential growth.

1. Introduction and main result

We study the existence of ground state to the Chern-Simons-Schrödinger system (CSS system) involving a nonlinearity $f(u)$ in the case of critical exponential growth

$$
-\Delta u + u + A_0 u + \sum_{j=1}^2 A_j^2 u = f(u),
$$

$$
\partial_1 A_0 = A_2 |u|^2, \quad \partial_2 A_0 = -A_1 |u|^2,
$$

$$
\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2} u^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0,
$$

where $A_\mu \in \mathbb{R}, \mu = 0, 1, 2,$ is vector potential of the gauge fields, $\partial_0 = \frac{\partial}{\partial t}, \partial_1 = \frac{\partial}{\partial x_1}, \partial_2 = \frac{\partial}{\partial x_2}$. This system arises in the study of the standing wave of Chern-Simons-Schrödinger system that describes the dynamics of large number of particles in an electromagnetic field. Chern-Simons terms in CSS system are necessary ingredients in various anyon models describing many fermion systems such as electron pairing in the high-temperature superconductor, fractional quantum Hall effect and Aharonov-Bohm scattering, see [28, 29] and references therein.

Since the gauge field $A_\mu$ is coupled to complex field $\phi \in \mathbb{C}$, the Euler-Lagrange equations of the energy which are given by

$$
i D_0 \phi + (D_1 D_1 + D_2 D_2) \phi = f(\phi),
$$

$$
\partial_0 A_1 - \partial_1 A_0 = -\text{Im}(\bar{\phi} D_2 \phi),
$$

$$
\partial_0 A_2 - \partial_2 A_0 = \text{Im}(\bar{\phi} D_1 \phi),
$$

$$
\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2} |\phi|^2.
$$

Here $D_\mu \phi = (\partial_\mu + i A_\mu) \phi, \mu = 0, 1, 2$. The CSS system (1.2) is invariant under the following gauge transformation $\phi \to e^{i \chi}, \quad A_\mu \to A_\mu - \partial_\mu \chi$ where $\chi : \mathbb{R}^{1+2} \to \mathbb{R}$ is an arbitrary $C^\infty$ function. We assume that the gauge field satisfies the Coulomb
gauge condition $\partial_0 A_0 + \partial_1 A_1 + \partial_2 A_2 = 0$. Then the standing wave $\psi(x, t) = e^{i\omega t} u$ satisfies
\[
-\Delta u + \omega u + A_0 u + A_1^2 u + A_2^2 u = f(u),
\]
\[
\partial_1 A_0 = A_2^2, \quad \partial_2 A_0 = -A_1^2, \quad \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2} |u|^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0.
\]

We say that $f(s)$ has subcritical growth at $+\infty$ if for all $\alpha > 0$,
\[
\lim_{s \to +\infty} \frac{f(s)}{e^{\alpha s^2}} = 0
\]
and $f(s)$ has critical growth at $+\infty$ if there exists $\alpha_0 > 0$ such that
\[
\lim_{s \to +\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0. \end{cases}
\]

We assume $f(u)$ satisfies the following conditions:

(A1) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(0) = 0$, $\lim_{s \to 0} F(s)/s^2 = 0$;

(A2) There exist $\theta > 6$ and $s_1 > 0$ such that for all $|s| \geq s_1$
\[
0 < \theta F(s) := \theta \int_0^s f(t) \, dt \leq sf(s);
\]

(A3) There exists $\beta_0 > 0$ such that
\[
\lim_{s \to +\infty} sf(s)e^{-\alpha_0 s^2} \geq \beta_0.
\]

We remark that the condition (A2) can be replaced by
\[
0 < F(s) \leq M_0 f(s), \quad \text{if } |s| \geq R_0,
\]
for some constants $R_0, M_0 > 0$.

The standing waves of (1.2) have been investigated by Byeon, Huh and Seok [2].

The standing waves of (1.2) have been investigated by Byeon, Huh and Seok [2].

They were seeking the radial solutions when $f(u) = \lambda |u|^{p-1} u$, $\lambda > 0$ and $p > 2$ by variational methods, see also [11, 12]. A series of existence and nonexistence results of solitary waves has been established in [4, 5, 17, 24, 25, 26, 30].

We studied the existence, non-existence, and multiplicity of standing waves to the nonlinear CSS systems with an external potential $V(x)$ without the Ambrosetti-Rabinowitz condition in [27].

Moreover, we have shown the existence of nontrivial solutions to Chern-Simons-Schrödinger systems (1.1) by using the concentration compactness principle with $V(x)$ is a constant and the argument of global compactness with $V \in C(\mathbb{R}^2)$ and $0 < V_0 < V(x) < V_\infty$ under the condition $p > 4$ in [28].

We also have obtained the concentration behavior of the solutions to system (1.1) with $p > 6$ in [29].

The main characteristic of system (1.1) is that the non-local term $A_\mu, \mu = 0, 1, 2$ depends on $u$ and there is a lack of compactness in $\mathbb{R}^2$. By using the variational method we can obtain the following result.

**Theorem 1.1.** If $f(s)$ is critical growth and (A1)–(A3) hold, then Problem (1.1) has a solution.
We mention that Zhang and Wan also proved that if $f(s)$ is subcritical growth then Problem (1.1) has a solution in $[33]$. On the other hand, radial solutions for the Chern-Simons-Schrödinger equation with exponential growth can be found in [16]. To demonstrate the desired result, we employ the approach which was developed by do Ó, Medeiros and Severo [8]. Here we mention that Pan, Li, Tang [23] studied CSS system with critical growth; see also [6, 21]. Sign-changing solutions have been found for the nonlinear Chern-Simons-Schrödinger equations in [31] Normalized solutions of Chern-Simons-Schrödinger system are studied by [10, 22, 32].

This article is organized as follows. In Section 2 we introduce the framework and prove some technical lemmas. In Section 3 we prove Theorem 1.1.

2. Mathematical framework

In this section, we outline the variational framework for a future study. We consider the functions which belong to the usual Sobolev space $H^1(\mathbb{R}^2)$ with

$$\|u\| = \left( \int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2 \, dx \right)^{1/2}.$$  

Define the functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla u|^2 + |u|^2 + A_1^2|u|^2 + A_2^2|u|^2 \right) \, dx - \int_{\mathbb{R}^2} F(u) \, dx,$$  

where $F(u) = \int_0^u f(s) \, ds$. We have the derivative of $J$ in $H^1(\mathbb{R}^2)$ as follows

$$\langle J'(u), \eta \rangle = \int_{\mathbb{R}^2} \left( \nabla u \nabla \eta + u \eta - f(u) \eta + (A_1^2(u) + A_2^2(u))u \eta + A_0u \eta \right) \, dx$$

$$+ 2 \int_{\mathbb{R}^2} A_1u^2 \int_{\mathbb{R}^2} K_2(x,y)u(y)\eta(y) \, dy \, dx$$

$$+ 2 \int_{\mathbb{R}^2} A_2u^2 \int_{\mathbb{R}^2} -K_1(x,y)u(y)\eta(y) \, dy \, dx,$$  

for all $\eta \in C_0^\infty(\mathbb{R}^2)$. Especially, from (2.4), we have

$$\langle J'(u), u \rangle = \int_{\mathbb{R}^2} \left( |\nabla u|^2 + |u|^2 + 3(A_1^2(u) + A_2^2(u))|u|^2 - f(u)u \right) \, dx.$$  

Substituting $\partial_1 A_0 = A_2 u^2$, $\partial_2 A_0 = -A_1 u^2$ in the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$, we obtain

$$0 = \partial_2 \partial_1 A_0 - \partial_1 \partial_2 A_0$$

$$= \partial_2 (A_2 u^2) + \partial_1 (A_1 u^2)$$

$$= 2u(A_1 \partial_1 u + A_2 \partial_2 u) + u^2(\partial_1 A_1 + \partial_2 A_2).$$

This implies

$$\sum_{j=1}^2 A_j \partial_j u = 0.$$  

This also implies the imaginary part of the CSS system vanishes.
Again we can derive from $\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2} u^2$ that
\[
\int_{\mathbb{R}^2} A_0 |u|^2 \, dx = -2 \int_{\mathbb{R}^2} A_0 (\partial_1 A_2 - \partial_2 A_1) \, dx
= 2 \int_{\mathbb{R}^2} (A_2 \partial_1 A_0 - A_1 \partial_2 A_0) \, dx
= 2 \int_{\mathbb{R}^2} (A_1^2 + A_2^2) |u|^2 \, dx.
\]

Combining the equation $\partial_1 A_2 - \partial_2 A_1 = -u^2/2$ and the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$ provides that the components $A_j$ can be determined from $u$ by solving elliptic system
\[
\Delta A_1 = \partial_2 (\frac{|u|^2}{2}), \quad \Delta A_2 = -\partial_1 (\frac{|u|^2}{2}).
\]
That are equivalent to
\[
\mathcal{F}(A_1) = -\frac{\xi_2}{|\xi|^2} \mathcal{F}(\frac{|u|^2}{2}), \quad \mathcal{F}(A_2) = \frac{\xi_1}{|\xi|^2} \mathcal{F}(\frac{|u|^2}{2})
\]
where $\mathcal{F}$ denotes the Fourier transform of an integrable function.

Then we have the following representation of $(A_1, A_2)$,
\[
A_1 = A_1(u) = K_2 * (\frac{|u|^2}{2}) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} \frac{|u|^2(y)}{2} \, dy,
\]
\[
A_2 = A_2(u) = -K_1 * (\frac{|u|^2}{2}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} \frac{|u|^2(y)}{2} \, dy,
\]
where $K_j = \frac{-x_j}{2\pi |x|^2}$ for $j = 1, 2$ and $*$ denotes the convolution. Moreover, the system $\partial_1 A_0 = A_2 u^2$, $\partial_2 A_0 = -A_1 u^2$ implies that
\[
\Delta A_0 = \partial_1 (A_2 |u|^2) - \partial_2 (A_1 |u|^2),
\]
which yields the following representation
\[
A_0 = A_0(u) = K_1 * (A_1 |u|^2) - K_2 * (A_2 |u|^2)
= K_1 * (|u|^2 K_2 * (\frac{|u|^2}{2}) + K_2 * (|u|^2 K_1 * (\frac{|u|^2}{2})).
\]

We know that $J$ is well defined in $H^1(\mathbb{R}^2)$, $J \in C^1(H^1(\mathbb{R}^2))$, and the weak solution of (1.1) is the critical point of the functional $J$ from the following properties, which we refer to [28 29].

**Proposition 2.1.** Let $1 < s < 2$ and $\frac{1}{s} - \frac{1}{q} = \frac{1}{2}$.

(i) There is a constant $C$ depending only on $s$ and $q$ such that
\[
\left( \int_{\mathbb{R}^2} |Tu(x)|^q \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^2} |u(x)|^s \, dx \right)^{1/s},
\]
where the integral operator $T$ is defined as
\[
Tu(x) := \int_{\mathbb{R}^2} \frac{u(y)}{|x - y|} \, dy.
\]
Proposition 2.3. Let \( u \in H^1(\mathbb{R}^2) \), then we have for \( j = 1, 2 \),
\[
\|A_j^2(u)\|_{L^4(\mathbb{R}^2)} \leq C\|u\|_{L^2(\mathbb{R}^2)}^2,
\]
\[
\|A_0(u)\|_{L^2(\mathbb{R}^2)} \leq C\|u\|_{L^2(\mathbb{R}^2)}^2\|u\|_{L^4(\mathbb{R}^2)}^2.
\]

(iii) For \( q' = \frac{n}{q-1} \) and \( j = 1, 2 \), we have
\[
\|A_j(u)\|_{L^2(\mathbb{R}^2)} \leq \|A_j(u)\|_{L^4(\mathbb{R}^2)}\|u\|_{L^{2q'}(\mathbb{R}^2)}^2.
\]

We will need the following properties of the convergence for \( A_j \), see [29].

Proposition 2.2. Suppose that \( u_n \) converges to \( u \) a.e. in \( \mathbb{R}^2 \) and \( u_n \) converges weakly to \( u \) in \( H^1(\mathbb{R}^2) \). Let \( A_{\mu,n} := A_\mu(u_n(x)) \), \( \mu = 0, 1, 2 \). Then
(i) \( A_{\mu,n} \) converges to \( A_\mu(u(x)) \) a.e. in \( \mathbb{R}^2 \).
(ii) \( \int_{\mathbb{R}^2} A_{j,n}^2 u_n dx \), \( \int_{\mathbb{R}^2} A_{j,n}^2 |u_n|^2 dx \), and \( \int_{\mathbb{R}^2} A_{2,n}^2 |u_n|^2 dx \) converge to \( \int_{\mathbb{R}^2} A_j^2 |u|^2 dx \), for \( j = 1, 2 \); \( \int_{\mathbb{R}^2} A_0 u_n dx \) and \( \int_{\mathbb{R}^2} A_0 |u_n|^2 dx \) converge to \( \int_{\mathbb{R}^2} A_0 |u|^2 dx \).
(iii) \( \int_{\mathbb{R}^2} |A_j(u_n - u)|^2 |u_n - u|^2 dx = \int_{\mathbb{R}^2} |A_j(u_n)|^2 |u_n|^2 dx - \int_{\mathbb{R}^2} |A_j(u)|^2 |u|^2 dx + o_n(1) \), for \( j = 1, 2 \).

To prove the mountain pass construction, we need the following results from [8].

Proposition 2.3. (i) If \( \alpha > 0 \) and \( u \in H^1(\mathbb{R}^2) \), then
\[
\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1) dx < \infty.
\]
Moreover, if \( \|\nabla u\|_2^2 \leq 1 \), \( \|u\|_2 \leq M < \infty \) and \( \alpha < 4\pi \) then there exists a constant \( C = C(M, \alpha) \), which depends only on \( M \) and \( \alpha \), such that
\[
\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1) dx < C(M, \alpha).
\]
(ii) Let \( \{w_n\} \) in \( H^1(\mathbb{R}^2) \) satisfy \( \|w_n\| = 1 \). Suppose that \( w_n \) weakly converges to \( w_0 \) in \( H^1(\mathbb{R}^2) \) with \( \|w_0\| < 1 \). Then for all \( 0 < \beta < \frac{4\pi}{1-\|w_0\|^2} \),
\[
\sup_n \int_{\mathbb{R}^2} (e^{\beta|w_n|^2} - 1) dx < \infty.
\]
(iii) Let \( \beta > 0 \) and \( r > 1 \). Then for each \( \alpha > r \) there exists a positive constant \( C = C(\alpha) \) such that for all \( s \in \mathbb{R} \),
\[
(e^{\beta s^2} - 1)^r \leq C(e^{\alpha s^2} - 1).
\]
In particular, if \( u \in H^1(\mathbb{R}^2) \) then \( (e^{\beta u^2} - 1)^r \) belongs to \( L^1(\mathbb{R}^2) \).
(iv) If \( v \in H^1(\mathbb{R}^2) \), \( \beta > 0 \), \( q > 0 \) and \( \|v\| \leq M \) with \( \beta M^2 < 4\pi \), then there exists \( C = C(\beta, M, q) > 0 \) such that
\[
\int_{\mathbb{R}^2} (e^{\beta v^2} - 1)|v|^q dx \leq C\|v\|^q.
\]

Next, we prove that the energy functional \( J \) has the mountain pass structure.

Lemma 2.4. Assume (A1), (A2), and \([15] \) hold. Then there exists \( \rho > 0 \) such that \( J(u) > 0 \) if \( \|u\| = \rho \).

Proof. From (A1), (A2), and \([15] \), there exists \( \epsilon < \lambda/2 \), where \( \lambda \) is the best constant of \( L^2(\mathbb{R}^2) \mapsto H^1(\mathbb{R}^2) \), such that
\[
|F(s)| \leq \epsilon|s|^2 + C_1|s|^q(e^{\alpha s^2} - 1),
\]
where \( \alpha \) is a positive constant.
Lemma 2.6. Assume\( \varepsilon > 0 \) where\( \varepsilon < \frac{1}{2} \lambda \) and \( q > 2 \), we can choose \( \rho > 0 \) such that for \( \|u\| = \rho \)
\[
J(u) \geq \left( \frac{1}{2} - \frac{\varepsilon}{\lambda} \right) \|u\|^2 - C_1 \|u\|^q.
\] (2.10)
Consequently, by using \( \varepsilon < \frac{1}{2} \lambda \) and \( q > 2 \), we can choose \( \rho > 0 \) such that for \( \|u\| = \rho \)
\[
J(u) \geq \|u\| \left[ \frac{1}{2} - \frac{\varepsilon}{\lambda} \right] \|u\| - c \|u\|^{q-1} > 0.
\] □

Lemma 2.5. Assume that \( f \) satisfies (A2). Then there exists \( e \in E \) with \( \|e\| > \rho \) such that \( I(e) < \inf_{\|u\| = \rho} I(u) \).

Proof. Let \( u \in H^1(\mathbb{R}^2) \) such that \( u \equiv s_1 \) in \( B_1 \), \( u \equiv 0 \) in \( B_2^c \) and \( u \geq 0 \). Denoting \( k = \text{supp}(u) \). From (A2), for all \( s \in \mathbb{R} \) we have
\[
F(s) \geq C_1 |s|^\theta - C_2.
\] (2.11)
Then, for \( t > 1 \) we have
\[
I(tu) \leq \frac{t^2}{2} \|u\|^2 + ct^6 \|u\|^6 - ct^6 \int_{\{x: |u(x)| \geq s_1\}} u^\theta \, dx + C_1 |k|.
\]
Since \( \theta > 6 \), we obtain \( I(tu) \to -\infty \) as \( t \to +\infty \). Setting \( e = tu \) with \( t \) large enough, the proof is complete. □

We need the following result to prove the (PS) condition.

Lemma 2.6. Assume (A2) and (1.5). Let \( (u_n) \) in \( E \) such that \( J(u_n) \to c \) and \( J'(u_n) \to 0 \). Then, \( \|u_n\| \leq c_0 \), \( \int_{\mathbb{R}^2} f(u_n)u_n \, dx \leq c_0 \), and \( \int_{\mathbb{R}^2} F(u_n) \, dx \leq c_0 \).

Proof. First, we prove that \( \|u_n\| \leq c_0 \). We have
\[
\frac{1}{2} \|u_n\|^2 + \frac{1}{2} \int_{\mathbb{R}^2} \left( A_{1,n}^2 |u_n|^2 + A_{2,n}^2 |u_n|^2 \right) \, dx - \int_{\mathbb{R}^2} F(u_n) \, dx = c + o_n(1)
\]
and for any \( \varphi \in E \),
\[
\int_{\mathbb{R}^2} (\nabla u_n \nabla \varphi + u_n \varphi) \, dx + \int_{\mathbb{R}^2} (A_{1,n}^2 + A_{2,n}^2 + A_{0,n}) u_n \varphi \, dx - \int_{\mathbb{R}^2} f(u_n) \varphi \, dx = o_n(\|\varphi\|).
\]
From (A2) and \( \theta > 6 \), we obtain
\[
\theta c + \varepsilon_n \|u_n\| \geq \left( \frac{\theta}{2} - 1 \right) \|u_n\|^2 + \left( \frac{\theta}{2} - 3 \right) \int_{\mathbb{R}^2} \left( A_{1,n}^2 |u_n|^2 + A_{2,n}^2 |u_n|^2 \right) \, dx
\]
\[
- \int_{\mathbb{R}^2} \left( \theta F(u_n) - f(u_n)u_n \right) \, dx \geq \left( \frac{\theta}{2} - 1 \right) \|u_n\|^2 - \int_{\{x: |u_n(x)| < s_1\}} \left( \theta F(u_n) - f(u_n)u_n \right) \, dx,
\]
where \( \varepsilon_n \to 0 \) as \( n \to \infty \). Using that \( |f(s)s - F(s)| \leq c_1 |s| \) for all \( |s| \leq s_1 \), we obtain
\[
\theta c + \varepsilon_n \|u_n\| \geq \left( \frac{\theta}{2} - 1 \right) \|u_n\|^2 - c_1 \|u_n\|,
\]
which implies \( \|u_n\| \leq c_0 \). Next, we show \( \int_{\mathbb{R}^2} f(u_n)u_n \, dx \leq c_0 \) and \( \int_{\mathbb{R}^2} F(u_n) \, dx \leq c_0 \). In fact, since \( \|u_n\| \leq c_0 \), \( J(u_n) \to c \), and \( J'(u_n) \to 0 \), we have

\[
\int_{\mathbb{R}^2} F(u_n) = \frac{1}{2} \|u_n\|^2 + \frac{1}{2} \int_{\mathbb{R}^2} (A_{1,n}^2|u_n|^2 + A_{2,n}^2|u_n|^2) \, dx \leq c + o_n(1),
\]

\[
\int_{\mathbb{R}^2} f(u_n)u_n \, dx = \|u_n\|^2 + 3 \int_{\mathbb{R}^2} (A_{1,n}^2 + A_{2,n}^2) u_n^2 \, dx + \varepsilon_n \|u_n\|,
\]

where \( \varepsilon_n \to 0 \) as \( n \to \infty \). By Proposition 2.1 and Sobolev embedding theorem, we obtain

\[
\int_{\mathbb{R}^2} F(u_n) \leq \frac{1}{2} \|u_n\|^2 + C\|u_n\|^4 - c + o_n(1),
\]

\[
\int_{\mathbb{R}^2} f(u_n)u_n \, dx = \|u_n\|^2 + C\|u_n\|^4 - \varepsilon_n \|u_n\|.
\]

From \( \|u_n\| \leq c_0 \), we obtain \( \int_{\mathbb{R}^2} f(u_n)u_n \, dx \leq c_0 \) and \( \int_{\mathbb{R}^2} F(u_n) \, dx \leq c_0 \). \( \square \)

### 3. PROOF OF MAIN RESULTS

First we need prove the Palais-Smale condition. Using Moser’s function sequences, we can obtain the minimax level of the mountain pass solution. Let

\[
\tilde{\psi}_n(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} 
(\log n)^{1/2} & \text{if } |x| \leq \frac{r_0}{n}, \\
\frac{\log|r_0/(|x|)}{(\log n)^{1/2}} & \text{if } \frac{r_0}{n} \leq |x| \leq r_0, \\
0 & \text{if } |x| > r_0.
\end{cases}
\]

Notice that \( \tilde{\psi}_n \in H^1(\mathbb{R}^2) \), supp \( \tilde{\psi}_n \subset \overline{B_{r_0}} \), for a fixed \( r_0 \). By using the fact

\[
\int_{\{a_0 < |x| < 1\}} \nabla \log |x| \, dx = 2\pi \int_{a_0}^1 |\nabla \log r|^2 r \, dr = 2\pi \int_{a_0}^1 r \, dr = -2\pi \ln a_0,
\]

we can prove that \( \int_{\mathbb{R}^2} |\nabla \tilde{\psi}_n|^2 \, dx = 1 \). Moreover,

\[
\int_{\mathbb{R}^2} |\tilde{\psi}_n|^2 \, dx = O\left(\frac{1}{\log n}\right), \quad \text{as } n \to \infty.
\]

Thus, we can conclude that \( \|\tilde{\psi}_n\| \to 1 \) as \( n \to \infty \).

Considering \( \psi_n = \tilde{\psi}_n / \|\tilde{\psi}_n\| \), we can rewrite

\[
\psi_n^2(x) = (2\pi)^{-1} \log n + d_n, \quad \text{for all } |x| \leq \frac{r_0}{n},
\]

where \( d_n = (2\pi)^{-1}(\|\tilde{\psi}_n\|^{-1} - 1) \log n \). Consequently

\[
\frac{d_n}{\log n} \to 0 \quad \text{as } n \to \infty.
\]

On the other hand, we know that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} |\psi_n|^2 \, dx = 0.
\]

By the Hölder inequality, for \( 2\theta + q_1(1 - \theta) = 4 \) we have

\[
\|\psi_n\|^4_{L^4(\mathbb{R}^2)} \leq \|\psi_n\|^2_{L^2(\mathbb{R}^2)} \|\psi_n\|^{(1-\theta)q_1}_{L^{q_1}(\mathbb{R}^2)}.
\]
Then we can deduce that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} |\psi_n|^{q_1} \, dx = 0 \quad \text{for } q_1 \geq 2,
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} A_j(\psi_n)^2 \psi_n^2 \, dx = 0.
\]

**Proposition 3.1.** Assume that (A2)–(A4), hold. Then there exists \( n \in \mathbb{N} \) such that
\[
\max_{t \geq 0} \left[ \frac{t^2}{2} + \frac{t^6}{2} \int_{\mathbb{R}^2} (A_1^2(\psi_n) + A_2^2(\psi_n)) |\psi_n|^2 \, dx \right] - \int_{\mathbb{R}^2} F(t\psi_n) \, dx \leq \frac{2\pi}{\alpha_0}.
\]

*Proof.* Let us choose \( r_0 > 0 \) such that
\[
\beta_0 > \frac{2}{r_0^2 \alpha_0},
\]
where \( \beta_0 \) has been fixed in (A3). Suppose by contradiction that for all \( n \)
\[
\frac{t^2}{2} + \frac{t^6}{2} \int_{\mathbb{R}^2} (A_1^2(\psi_n) + A_2^2(\psi_n)) |\psi_n|^2 \, dx - \int_{\mathbb{R}^2} F(t\psi_n) \, dx \geq \frac{2\pi}{\alpha_0}.
\]

From (A2), there exist positive constants \( C_1, C_2 \) such that \( F(s) \geq C_1 e^{|s|^2} - C_2 \).

Consequently, if \( t > 0 \) is sufficiently large and \( m > 2 \), we have
\[
\int_{\mathbb{R}^2} F(t\psi_n) \, dx \geq -C_1 + \int_{\{\psi_n \geq s_1\}} e^{t\psi_n/M_0} \, dx
\]
\[
\geq -C_1 + C_3 \int_{\{\psi_n \geq s_1\}} (\psi_n)^m \, dx
\]
\[
\geq -C_1 + C_3 t^m \int_{\{\psi_n \geq s_1\}} (\psi_n)^m \, dx.
\]

Hence, for each \( n \) there exists unique maximum point \( t_n \) such that
\[
\frac{t_n^2}{2} + \frac{t_n^6}{2} \int_{\mathbb{R}^2} (A_1^2(\psi_n) + A_2^2(\psi_n)) |\psi_n|^2 \, dx - \int_{\mathbb{R}^2} F(t_n\psi_n) \, dx = \max_{t > 0} J(t\psi_n)
\]
and
\[
\frac{d}{dt} J(t\psi_n) \bigg|_{t=t_n} = 0.
\]

From which it follows that
\[
\frac{t_n^2}{2} + 3t_n^6 \int_{\mathbb{R}^2} (A_1^2(\psi_n) + A_2^2(\psi_n)) |\psi_n|^2 \, dx - \int_{\mathbb{R}^2} t_n \psi_n f(t_n\psi_n) \, dx = 0.
\]

By (A3) for each \( \varepsilon > 0 \) there exists \( R_\varepsilon > 0 \) such that
\[
\psi_n f(\psi_n) \geq (\beta_0 - \varepsilon) \exp(\alpha_0 \psi_n^2)
\]
for all \( \psi_n \geq R_\varepsilon \) and \( |x| \leq r_0 \). From (3.4) and (3.5), we have
\[
\frac{t_n^2}{2} + 3t_n^6 \int_{\mathbb{R}^2} (A_1^2(\psi_n) + A_2^2(\psi_n)) |\psi_n|^2 \, dx
\]
\[
\geq (\beta_0 - \varepsilon) \pi \left( \frac{r_0}{n} \right)^2 \exp(\frac{\alpha_0}{2\pi} t_n^2 \log n + 2\alpha_0 t_n^2 d_n).
\]

That is,
\[
1 + 3t_n^4 \int_{\mathbb{R}^2} (A_1^2(\psi_n) + A_2^2(\psi_n)) |\psi_n|^2 \, dx
\]
In fact, by (3.3), (3.4), and (A2), we have
\[ t_n^2 \geq (\beta_0 - \epsilon)\pi r_0^2 \exp\left(\frac{\alpha_0}{4\pi} t_n^2 - 2 \log t_n - 2 \log n\right). \]

Since \( \int_{\mathbb{R}^2} (A_1^2(\psi_n) + A_2^2(\psi_n))|\psi_n|^2 \, dx \to 0 \), as \( n \to \infty \), we obtain that \( \{t_n\} \) is bounded.

We claim that
\[ t_n^2 \rightarrow \frac{4\pi}{\alpha_0}, \quad \text{as} \quad n \to \infty. \] (3.7)

In fact, by (3.3), (3.4), and (A2), we have
\[ t_n^2 + \frac{t_n^6}{2} \int_{\mathbb{R}^2} (A_1^2(\psi_n) + A_2^2(\psi_n))|\psi_n|^2 \, dx \geq \frac{2\pi}{\alpha_0} + \int_{\{t_n\psi_n \leq 1\}} F(t_n\psi_n) \, dx \]

Since \( \{t_n\} \) is bounded, by (2.9) and \( \|\tilde{\psi}_n\|^2 \to 1 \) as \( n \to \infty \), we obtain
\[ \left| \int_{\{t_n\psi_n \leq 1\}} F(t_n\psi_n) \, dx \right| \leq C \int_{\mathbb{R}^2} \psi_n^2 \, dx = C \frac{1}{\|\psi_n\|^2} \int_{\mathbb{R}^2} \tilde{\psi}_n^2 \, dx \to 0. \]

Note that \( \int_{\mathbb{R}^2} (A_1^2(\psi_n) + A_2^2(\psi_n))|\psi_n|^2 \, dx \to 0 \) as \( n \to \infty \). Consequently,
\[ t_n^2 \geq \frac{4\pi}{\alpha_0} + o_n(1), \quad \text{as} \quad n \to \infty. \]

Suppose by contradiction that \( \lim_{n \to \infty} t_n^2 > \frac{4\pi}{\alpha_0} \). From (3.6), we have
\[ t_n^2 + 3t_n^6 \int_{\mathbb{R}^2} (A_1^2(\psi_n) + A_2^2(\psi_n))|\psi_n|^2 \, dx \geq (\beta_0 - \epsilon)\pi r_0^2 \exp\left(\frac{\alpha_0}{4\pi} t_n^2 - 1\right)2 \log n + 2\alpha_0^2 d_n. \]

Since (3.4), the last inequality contradicts the boundedness of \( \{t_n\} \) and the claim holds.

Let us denote
\[ \Omega_{1,n} := \{ x \in B_{r_0} : t_n\psi_n \geq R_e \}, \quad \text{and} \quad \Omega_{2,n} := B_{r_0} \setminus \Omega_{1,n}. \]

By (3.4) and (3.5), we obtain
\[ t_n^2 + 3t_n^6 \int_{\mathbb{R}^2} (A_1^2(\psi_n) + A_2^2(\psi_n))|\psi_n|^2 \, dx \geq (\beta_0 - \epsilon) \int_{B_{r_0}} e^{\alpha_0 t_n^2 \psi_n^2} + \int_{\Omega_{2,n}} t_n \psi_n f(t_n \psi_n) - (\beta_0 - \epsilon) \int_{\Omega_{2,n}} e^{\alpha_0 t_n^2 \psi_n^2}. \] (3.8)

Since \( \psi_n(x) \to 0 \) as \( n \to \infty \) and the characteristic functions \( \chi_{\Omega_{2,n}} \to 1 \) for almost every \( x \) such that \( |x| \leq r \). By the Lebesgue dominated convergence theorem, we have
\[ \int_{\Omega_{2,n}} t_n \psi_n f(t_n \psi_n) \, dx \to 0 \quad \text{and} \quad \int_{\Omega_{2,n}} e^{\alpha_0 t_n^2 \psi_n^2} \, dx \to \pi r_0^2 \quad \text{as} \quad n \to \infty. \]

By \( t_n^2 \geq \frac{4\pi}{\alpha_0} \), we obtain
\[ \int_{\{|x| \leq r_0\}} e^{\alpha_0 t_n^2 \psi_n^2} \, dx \geq \int_{\{|x| \leq r_0\}} e^{4\pi \psi_n^2} \, dx = \int_{\{|x| \leq \frac{r_0}{4}\}} e^{4\pi \psi_n^2} \, dx + \int_{\{|x| \leq \frac{r_0}{4} \leq |x| \leq r_0\}} e^{4\pi \psi_n^2} \, dx. \] (3.9)
A direct computation gives
\[
\lim_{n \to \infty} \int_{\{|x| \leq \frac{r_0}{n}\}} e^{4\pi \psi^2} \, dx = \lim_{n \to \infty} \int_{\{|x| \leq \frac{r_0}{n}\}} e^{2 \log n + 4\pi \psi^2} \, dx
\]
\[
= \lim_{n \to \infty} \frac{\pi r_0^2}{n^2} n^2 + 4\pi (\log n)^{-1} d_n = \pi r_0^2.
\]
Set \( t = \log(\frac{r_0}{|x|})/(\xi_n \log n) \), where \( \xi_n = \|\tilde{\psi}_n\| > 1 \). We have
\[
\int_{\{|\frac{r_0}{n}\leq|x|\leq r_0\}} e^{4\pi \psi^2} \, dx = \frac{2\pi r_0^2}{\xi_n \log n} \int_0^{\xi_n^{-1}} e^{2 \log n(t^2 - \xi_n t)} \, dt.
\]
Since
\[
t^2 - \xi_n t \geq \begin{cases} -\xi_n t & \text{if } 0 \leq t \leq \frac{\xi_n^{-1}}{2}, \\ (2\xi_n^{-1} - \xi_n)(t - \xi_n^{-1}) + (\xi_n^{-2} - 1) & \text{if } \frac{\xi_n^{-1}}{2} \leq t \leq \xi_n^{-1}, \end{cases}
\]
we obtain
\[
\lim_{n \to \infty} \int_{\{|\frac{r_0}{n}\leq|x|\leq r_0\}} e^{4\pi \psi^2} \, dx \geq 2\pi r_0^2.
\]
Taking \( n \to \infty \) in (3.8) and using (3.7), we obtain
\[
\frac{4\pi}{\alpha_0} \geq (\beta_0 - \varepsilon)2\pi r_0^2,
\]
which gives \( \beta_0 \leq \frac{2}{\alpha_0 r_0^2} \). This contradicts (3.2). The proof is complete. \( \square \)

Assuming that
\[
\lim_{n \to \infty} \|u_n\|^2 < \frac{4\pi}{\alpha_0},
\]
then there exists a subsequence of \( \{u_n\} \) which converges to \( u_0 \) in \( H^1(\mathbb{R}^2) \).

**Proposition 3.2.** \( J(u_0) = c \).

**Proof.** Since \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^2) \), there exists a subsequence denoted again by \( \{u_n\} \) such that
\[
u_n \to u_0 \quad \text{in} \quad H^1(\mathbb{R}^2),
\]
\[
u_n \to u_0 \quad \text{in} \quad L^q_{\text{loc}}(\mathbb{R}^2), \quad q \geq 1,
\]
\[
u_n(x) \to u_0(x) \quad \text{a.e. in} \quad \mathbb{R}^2.
\]
Moreover, for any \( R > 0 \),
\[
\lim_{n \to \infty} \int_{B_R} (F(u_n) - F(u_0)) \, dx = 0.
\]
It is known that for \( u \in L^2(\mathbb{R}^2) \), the Schwartz symmetrization of \( u \) satisfies
\[
|u^*| \leq \|u^*\|_{L^2(\mathbb{R}^2)} \frac{1}{\sqrt{\pi |x|}}.
\]
Since
\[
\int_{B_R^c} F(u_n) \leq C_1 \int_{B_R^c} |u_n|^2 + C_2 \int_{B_R^c} (|u_n| e^{|u_n|^2} - 1) \, dx,
\]
Therefore, there exists \( \delta > 0 \) such that
\[
\max \left\{ \int_{B_R^c} F(u_n) \, dx, \int_{B_R^c} F(u_0) \, dx, \int_{B_R^c} (F(u_n) - F(u_0)) \, dx \right\} \leq \frac{\delta}{3},
\]
from which it follows that
\[
\int_{\mathbb{R}^2} (F(u_n) - F(u_0)) \, dx < \delta.
\]
Hence by using \( J(u_n) \to c \) we conclude that
\[
\frac{1}{2} \left\| u_n \right\|^2 + \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u_n) + A_2^2(u_n))|u_n|^2 \, dx = c + \int_{\mathbb{R}^2} F(u_0) \, dx + o_n(1).
\]
We observe that \( \lim_{n \to \infty} \| u_n \| \geq \| u_0 \| > 0 \), so that we define
\[
w_n = \frac{u_n}{\| u_n \|} \quad \text{and} \quad w_0 = \frac{u_0}{\lim_{n \to \infty} \| u_n \|}.
\]
Then \( \| w_n \| = 1 \) and \( w_n \rightharpoonup w_0 \) in \( H^1(\mathbb{R}^2) \). Suppose that \( \| w_0 \| < 1 \). By Proposition 3.1 we see that \( \alpha_0 < \frac{2\pi}{c - J(u_0)} \). Let us choose \( \beta > 1 \) sufficiently close to 1 and \( \delta > 0 \) such that
\[
\beta \alpha_0 \| u_n \|^2 \leq \frac{2\pi \| u_n \|^2}{c - J(u_0)} - \delta
\]
\[
\leq 4\pi \frac{c + \int_{\mathbb{R}^2} F(u_0) \, dx - \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u_0) + A_2^2(u_0))|u_0|^2 \, dx + o_n(1)}{c - J(u_0)} - \delta.
\]
On the other hand, by using the formula for \( J(u_0) \) and \( J(u_0) < c \) we deduce that
\[
(1 - \| w_0 \|^2)(c + \int_{\mathbb{R}^2} F(u_0) \, dx - \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u_0) + A_2^2(u_0))|u_0|^2 \, dx + o_n(1))
\]
\[
\leq c + \int_{\mathbb{R}^2} F(u_0) \, dx - \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u_0) + A_2^2(u_0))|u_0|^2 \, dx
\]
\[
- \| w_0 \|^2 \left( \int_{\mathbb{R}^2} F(u_0) \, dx - \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u_0) + A_2^2(u_0))|u_0|^2 \, dx + o_n(1) \right)
\]
\[
= c + (-J(u_0) + \frac{1}{2} \| w_0 \|^2)
\]
\[
- \| w_0 \|^2 \left( c + \int_{\mathbb{R}^2} F(u_0) \, dx - \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u_0) + A_2^2(u_0))|u_0|^2 \, dx + o_n(1) \right)
\]
\[
\leq c - J(u_0).
\]
Therefore, there exists \( \delta > 0 \) such that
\[
\beta \alpha_0 \| u_n \|^2 \leq \frac{4\pi}{1 - \| w_0 \|^2} - \delta.
\]
Thus, \( (\beta + \epsilon) \alpha_0 \| u_n \|^2 \leq \frac{4\pi}{1 - \| w_0 \|^2} \), which implies by (ii) of Proposition 2.3 that
\[
\int_{\mathbb{R}^2} \left( e^{(\beta + \epsilon) \alpha_0 \| u_n \|^2 \delta} - 1 \right) \, dx \leq C.
\]
We observe that
\[ | \int_{\mathbb{R}^2} f(u_n)(u_n - u_0) \, dx | \leq | \int_{\mathbb{R}^2} (u_n - u_0) e^{\alpha n} \|u_n\|^2 \, dx | \leq C \int_{\mathbb{R}^2} |u_n - u_0|^\gamma d\nu. \]

Thus,
\[ \lim_{n \to \infty} \int_{\mathbb{R}^2} \nabla u_0 \nabla (u_n - u_0) + u_0 (u_n - u_0) \, dx = 0. \]

Hence, \( \{u_n\} \) converges to \( u_0 \) in \( H^1(\mathbb{R}^2) \).

\( \square \)

**Proof of Theorem 1.1**

Let \( \{u_n\} \) satisfying \( J(u_n) \to c_0 \) and \( J'(u_n) \to 0 \) as \( n \to \infty \).

By Lemma 2.6, \( \{u_n\} \) is bounded, up to a subsequence, we may assume that \( u_n \to u_0 \) in \( H^1(\mathbb{R}^2) \), \( u_n \to u_0 \) in \( L^q_{loc}(\mathbb{R}^2) \) for all \( q \geq 2 \) and \( u_n \to u_0 \) almost everywhere in \( \mathbb{R}^2 \), as \( n \to \infty \). Then, if \( f(s) \) satisfies (1.5), we have for each \( \alpha > \alpha_0 \) there exist \( b_1, b_2 > 0 \) such that for all \( s \in \mathbb{R} \), for all \( \alpha > 0 \),
\[ |f(s)| \leq b_1 |s| + b_2 (e^{\alpha s^2} - 1). \]

If the vanishing case occurs, then
\[ \lim_{n \to \infty} \int_{\mathbb{R}^2} |u_n|^2 \, dx = 0. \]

Consequently, by (3.10), (3.11), Hölder’s inequality, and Proposition 2.3, we have
\[ \lim_{n \to \infty} \int_{\mathbb{R}^2} |f(u_n)u_n| \, dx \]
\[ \leq \lim_{n \to \infty} \int_{\mathbb{R}^2} \left( b_1 |u_n|^2 + b_2 |u_n|(e^{\alpha |u_n|^2} - 1) \right) \, dx \]
\[ \leq b_1 \lim_{n \to \infty} \int_{\mathbb{R}^2} |u_n|^2 \, dx \]
\[ + b_2 \left( \lim_{n \to \infty} \int_{\mathbb{R}^2} |u_n|^2 \, dx \right)^{1/2} \left( \lim_{n \to \infty} \int_{\mathbb{R}^2} (e^{\alpha |u_n|^2} - 1)^2 \, dx \right)^{1/2} = 0. \]

By Proposition 2.2, we have
\[ \lim_{n \to \infty} \int_{\mathbb{R}^2} (A^2_1(u_n) + A^2_2(u_n)) u_n^2 \, dx = 0. \]

From (2.3), (3.12), (3.13), and that \( \{u_n\} \) is bounded, we have
\[ \|u_n\|^2 \]
\[ = \langle J'(u_n), u_n \rangle - 3 \int_{\mathbb{R}^2} ((A^2_1(u_n) + A^2_2(u_n)) u_n^2 \, dx + \int_{\mathbb{R}^2} f(u_n) u_n \, dx \to 0, \]

as \( n \to \infty \). By (2.9), (3.12), (3.14), and Hölder’s inequality, we have
\[ \lim_{n \to \infty} \int_{\mathbb{R}^2} F(u_n) \, dx \]
\[ \leq \lim_{n \to \infty} (\varepsilon \int_{\mathbb{R}^2} |u_n|^2 \, dx + C_1 \int_{\mathbb{R}^2} |u_n|^q (e^{\alpha |u_n|^2} - 1) \, dx) = 0. \]

This implies that \( 0 < J(u_n) \to 0 \) as \( n \to \infty \), which means that vanishing is impossible.

Hence only the nonvanishing case happens. Since
\[ \int_{\mathbb{R}^2} u^2(x) (A^0_1(u), \eta) \, dx \]
\[
\begin{align*}
&= \int_{\mathbb{R}^2} u^2(x) \left( \int_{\mathbb{R}^2} \frac{x_1 - y_1}{2\pi|x - y|^2} u(y)\eta(y)(y)A_2(u(y)) \, dy \\
&\quad - \int_{\mathbb{R}^2} \frac{x_2 - y_2}{2\pi|x - y|^2} u(y)\eta(y)A_1(u(y)) \, dx \\
&= \int_{\mathbb{R}^2} A_2(u(y))^2 u(y)\eta(y) \int_{\mathbb{R}^2} \frac{x_1 - y_1}{2\pi|x - y|^2} u(x)^2 \, dx \, dy \\
&\quad - \int_{\mathbb{R}^2} A_1(u(y))^2 u(y)\eta(y) \int_{\mathbb{R}^2} \frac{x_2 - y_2}{2\pi|x - y|^2} u^2(x) \, dx \, dy \\
&= \int_{\mathbb{R}^2} |A_2(u(y))|^2 u(y)\eta(y) + |A_1(u(y))|^2 u(y)\eta(y) \, dy,
\end{align*}
\]

For each \( \eta \in C_0^\infty(\mathbb{R}^2) \), we have
\[
0 = \lim_{n \to \infty} \langle J'(u_n), \eta \rangle = \lim_{n \to \infty} \int_{\mathbb{R}^2} \left( \nabla u_n \nabla \eta + u_n \eta + (A_1^2(u_n) + A_2^2(u_n)) u_n \eta + A_0(u_n)u_n \eta - f(u_n)\eta \right) \, dx = \langle J'(u_0), \eta \rangle.
\]

Hence \( u_0 \) is a weak solution of Problem (1.1). \( \square \)

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