P-MEAN \((\mu_1, \mu_2)\)-PSEUDO ALMOST PERIODIC PROCESSES AND APPLICATION TO INTEGRO-DIFFERENTIAL STOCHASTIC EVOLUTION EQUATIONS

MOEZ AYACHI, SYED ABBAS

Abstract. In this article, we investigate the existence and stability of p-mean \((\mu_1, \mu_2)\)-pseudo almost periodic solutions for a class of non-autonomous integro-differential stochastic evolution equations in a real separable Hilbert space. Using stochastic analysis techniques and the contraction mapping principle, we prove the existence and uniqueness of p-mean \((\mu_1, \mu_2)\)-pseudo almost periodic solutions. We also provide sufficient conditions for the stability of these solutions. Finally, we present three examples with numerical simulations to illustrate the significance of the main findings.

1. Introduction

The concept of almost periodic functions was introduced in 1923 by Harald Bohr [13]. It plays an important role in describing the phenomena that are more or less periodic, which can be observed frequently in many fields, such as biology, celestial mechanics, dynamical population, engineering, and so on. For more details, we refer to [2, 19] and references therein. Almost periodic solutions refer to solutions of differential equations that oscillate over time, but not in a strictly periodic manner. In other words, their oscillations are not exactly periodic, but they exhibit some sort of repetitive behavior. Since the introduction of almost periodicity, several extensions of this concept have been introduced, including pseudo-almost periodicity by Zhang [39], weighted pseudo-almost periodicity (WPAP) by Diagana [20, 21], and \(\mu\)-pseudo almost periodicity (PAP) by Blot et al. [12]. A more general-class pseudo-almost periodicity called \((\mu_1, \mu_2)\)-pseudo-almost periodicity was considered by Diagana et al. (see [22]). For more details about the \((\mu_1, \mu_2)\)-p.a.p. functions and their applications in the qualitative theory of differential equations, we refer the reader to [3, 5, 22, 30, 31, 35, 37, 38].

Further, stochastic perturbations are unavoidable and omnipresent in both real and artificial systems. Accordingly, investigating the dynamical behaviors of the systems described by various types of stochastic perturbations is highly important. For more information about the elementary theories for stochastic differential equations, we refer to [25, 32]. The concept of almost periodicity is important in...
probability for investigating stochastic processes. Recently, Bezandry and Diagana initiated the concept of $p$-mean almost periodic processes and applied it to the study of the existence and uniqueness of square-mean almost periodic mild solutions to some classes of stochastic differential equations \[8, 9, 11\]. Since then, the study of the existence and uniqueness of square-mean almost periodic mild solutions initiated the concept of $p$-mean almost periodic processes and applied it to the probability for investigating stochastic processes. Recently, Bezandry and Diagana introduced the concept of $p$-mean pseudo-almost periodicity by Ch’erif \[16\], $p$-mean $\mu$-pseudo almost periodicity by Diop et al. \[23\], and $(\mu_1, \mu_2)$-pseudo almost periodicity by Belmabrouk et al. \[4\]. Several works have focused on investigating square-mean almost periodic processes, their various extensions, and their applications in stochastic differential equations, we refer to \[7, 10, 14, 15, 27, 36, 40\].

Stochastic integro-differential equations play a crucial role in the qualitative theory of differential equations due to their application in engineering, dynamical population, neural networks, biology, and so on. As a direct consequence, they have attracted an increasing amount of attention over the past few years. This article deals with the $p$-mean $(\mu_1, \mu_2)$-pseudo almost periodic (\((\mu_1, \mu_2)\)-s.p.a.p. for short) mild solutions of the following non-autonomous integro-differential stochastic evolution equation in a real separable Hilbert space $\mathcal{E}$:

\[
\begin{align*}
Z'(t) &= A(t)Z(t) + F_1(t, Z(t)) + \int_{-\infty}^t Q(t-\zeta)F_2(\zeta, Z(\zeta))d\zeta \\
&\quad + \int_{-\infty}^t R(t-\zeta)G(\zeta, Z(\zeta))d\mathcal{W}(\zeta), \quad \forall t \in \mathbb{R},
\end{align*}
\]  

(1.1)

where $A(t) : D(A(t)) \subset L^p(\mathcal{P}, \mathcal{E}) \rightarrow L^p(\mathcal{P}, \mathcal{E})$ is a family of densely defined closed linear operators satisfying the so-called “Acquistapace-Terrani” conditions, $Q$ and $R$ are convolution type kernels in $L^1(0, +\infty)$ and $L^2(0, +\infty)$, respectively, satisfying \[24\] Assumption 3.2, $(\mathcal{W}(t) : t \in \mathbb{R})$ is a $\mathcal{K}$-valued $\mathcal{Q}$-Brownian motion. Here $F_1, F_2 : \mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}) \rightarrow L^p(\mathcal{P}, \mathcal{E})$ and $G : \mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}) \rightarrow L^p(\mathcal{P}, L^2_0)$ are jointly continuous functions satisfying some additional conditions. The spaces $L^p(\mathcal{P}, \mathcal{E})$, $L^2_0$, and the $\mathcal{Q}$-Brownian motion are defined in the next section.

Equation (1.1) was studied in several special cases. For instance, Bezandry \[6\] investigated the existence and uniqueness of square-mean almost periodic mild solutions for equation (1.1) for $p = 2$ and $A(t) = A$. Li \[27\] addressed the problem of the existence, uniqueness, and asymptotic stability of square-mean almost periodic mild solutions of equation (1.1) in the case $p = 2$. More recently, Mbaye \[29\] considered the problem of the existence of square-mean $\mu$-pseudo almost periodic mild solutions of equation (1.1) when $A(t) = A$. However, to the best of our knowledge, the existence, uniqueness, and stability of $p$-mean $(\mu_1, \mu_2)$-s.p.a.p. mild solutions of equation (1.1) is an untreated topic, which is the main motivation of this work.

Acquistapace-Terrani, which is discussed in this work, is an important condition that ensures the existence of a unique evolution family that is necessary for the corresponding integral form of a given differential equation. The equation considered in this work is very general in nature, and several other equations can be derived as special cases of it. Moreover, the concept of $(\mu_1, \mu_2)$-pseudo almost periodicity is a very general concept and can be applicable to situations where other kinds of functions, such as almost periodic and pseudo-almost periodic, cannot be used to describe the underlying dynamics. The conditions obtained for existence are very general in nature, and other conditions can be obtained as a special case. For
example, one can choose the second component as zero in order to get the corresponding result for almost periodic. Similar results can be obtained by adjusting the equation and space. Moreover, stability is also established, and the condition obtained holds for any $p$; the particular case when $p = 2$ is given special emphasis. Application to various fields along with numerical graphs makes this work more useful for researchers, especially those who are interested in application.

This work is structured as follows. In Section 2, we present some basic notations and definitions. In Section 3, we establish some sufficient conditions to support the existence, uniqueness, and stability of the p-mean $(\mu_1, \mu_2)$-s.p.a.p. mild solution on $R$ of equation (1.1). In the last section, three examples with numerical simulations are presented for better illustrations and to validate the analytical findings.

2. Preliminaries

This section introduces relevant notation, definitions and preliminary facts that are needed for the study.

2.1. $Q$-Brownian motion. Let $\beta_n(t)$, $n = 1, 2, 3, \ldots$ be a sequence of real valued standard Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathcal{P})$. Set $W(t) := \sum_{n=1}^{\infty} \sqrt{\xi_n} \beta_n(t) e_n$, $t \geq 0$, where $\xi_n \geq 0$ ($n \geq 1$), are non-negative real numbers and $(e_n)_{n \geq 1}$ is an orthonormal basis in the Hilbert space $K$. Let $Q$ be a non-negative symmetric operator with finite trace defined by $Q(e_n) = \xi_n e_n$, such that $\text{Tr}[Q] := \sum_{n=1}^{\infty} \xi_n < \infty$. It is well known that $E[W(t)] = 0$ and, for all $t \geq s \geq 0$, the distribution of $W(t) - W(s)$ is a Gaussian distribution $\mathcal{N}(0, (t-s)Q)$. The above mentioned $K$-valued stochastic processes $(W(t))_{t \geq 0}$ is called $Q$-Brownian motion. Note that a $K$-valued $Q$-Brownian motion $(W(t))_{t \in \mathbb{R}}$ can be obtained as follows: let $\{W_i(t) : t \in \mathbb{R}_+, i = 1, 2\}$ be independent $K$-valued $Q$-Brownian motion. Then

$$W(t) = \begin{cases} W_1(t), & \text{if } 0 \leq t, \\ W_2(-t), & \text{if } 0 \geq t, \end{cases}$$

is a $K$-valued $Q$-Brownian motion with $\mathbb{R}$ as time parameters. Let $\mathcal{F}_t = \sigma\{W(u) : u \leq t\}$.

To define stochastic integrals with respect to the $Q$-Brownian motion $W$, let us denote $K_0 = Q^{\frac{1}{2}}K$. Now define $L_0^2 := \{\varphi \in \mathcal{L}(K_0, \mathcal{E}) : \text{Tr}[\varphi Q \varphi^*] < \infty\}$ the space of Hilbert-Schmidt operators from $K_0$ to $\mathcal{E}$ equipped with the norm $\|\varphi\|_{L_2}^2 := \text{Tr}[\varphi Q \varphi^*]$ for any $\varphi \in L_0^2$. The next result is a particular case of [33, Lemma 2.2].

**Lemma 2.1** (\cite{11, 33}). For any $p \geq 2$ and for any arbitrary $L_0^2$-valued predictable processes $\Psi(t)$, $t \in [0, T]$, there exists a constant $C_p > 0$ such that

$$E\left[ \sup_{s \in [0,t]} \left\| \int_0^s \Psi(s) dW(s) \right\|^p \right] \leq C_p E\left[ \int_0^t \left\| \Psi(s) \right\|_{L_2}^2 ds \right]^{p/2}.$$ 

2.2. $P$-mean almost periodic stochastic processes. Assume that $(\mathcal{E}, \| \cdot \|)$ and $(K, \| \|_K)$ are real separable Hilbert spaces. $(\Omega, \mathcal{F}, \mathcal{P})$ is supposed to be a complete probability space. Let $p \geq 2$, and denote $L^p(\mathcal{P}, \mathcal{E})$ as the collection of all strongly measurable $\mathcal{E}$-valued random variables $Y$ satisfying $E\|Y\|^p < +\infty$, where the expectation $E$ is defined by $E[Y] := \int_{\Omega} Y(\omega) d\mathcal{P}(\omega)$. Note that $L^p(\mathcal{P}, \mathcal{E})$ is a Banach space when it is equipped with a norm $\|Y\|_{L_p} := [E\|Y\|^p]^{1/p}$. 

Definition 2.2 \((\text{III})\). A stochastic processes \(Z : \mathbb{R} \rightarrow L^p(\mathcal{P}, \mathcal{E})\) is said to be stochastically bounded if there exists a constant \(C > 0\) such that \(\mathbb{E}\|Z(t)\|^p \leq C\), for all \(t \in \mathbb{R}\). The process \(Z\) is said to be stochastically continuous if \(\lim_{t \to s} \mathbb{E}\|Z(t) - Z(s)\|^p = 0\).

We denote \(\text{SBC} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\) the collection of all stochastically bounded and continuous processes from \(\mathbb{R}\) into \(L^p(\mathcal{P}, \mathcal{E})\). Then \(\text{SBC} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\), \(\| \cdot \|_\infty\) is a Banach space, where

\[
\|Z\|_\infty := \sup_{t \in \mathbb{R}} [\mathbb{E}\|Z(t)\|^p]^{1/p}.
\]

Definition 2.3 \((\text{II})\). A continuous stochastic processes \(Z : \mathbb{R} \rightarrow L^p(\mathcal{P}, \mathcal{E})\) is said to be \(p\)-mean almost periodic processes, if for any \(\epsilon > 0\) we can find \(l = l(\epsilon) > 0\) such that for all \(r \in \mathbb{R}\), there exists \(r \in [0, \theta + l]\) satisfying

\[
\mathbb{E}\|Z(t + r) - Z(t)\|^p < \epsilon, \quad \forall t \in \mathbb{R}.
\]

We denote the collection of all such stochastic processes by \(\text{SAP} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\).

Proposition 2.4 \((\text{II})\). The following properties hold for the stochastic processes \(\text{SAP} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\):

1. \(\text{SAP} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})), \| \cdot \|_\infty\) is a Banach space.
2. \(\text{SAP} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\) is invariant by translation.
3. \(\text{SAP} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})) \subset \text{SBC} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\) is a closed subspace.

2.3. \(p\)-mean \((\mu_1, \mu_2)\)-pseudoo almost periodic processes. Let \(\mathcal{B}\) be the Lebesgue \(\sigma\)-field of \(\mathbb{R}\) and \(\mathfrak{M}\) be the set of all positive measures \(\mu\) on \(\mathcal{B}\) satisfying \(\mu(\mathbb{R}) = +\infty\) and \(\mu([r, s]) < +\infty\) for any \(r, s \in \mathbb{R}\) with \(r < s\).

To establish our results, we need the following assumptions:

(A1) Let \(\mu_1, \mu_2 \in \mathfrak{M}\) such that \(\limsup_{n \to +\infty} \frac{\mu_1([-m, m])}{\mu_2([-m, m])} < +\infty\).

(A2) For all \(s \in \mathbb{R}\), there exists \(\alpha > 0\) and a bounded interval \(I\) of \(\mathbb{R}\) such that

\[
\mu_1 \left( \{ c + s : c \in C \} \right) \leq \alpha \mu_1(C), \quad C \in \mathcal{B} \text{ satisfies } C \cap I = \emptyset.
\]

Definition 2.5. Let \(\mu_1, \mu_2 \in \mathfrak{M}\). A stochastic processes \(Z : \mathbb{R} \rightarrow L^p(\mathcal{P}, \mathcal{E})\) is said to be \(p\)-mean \((\mu_1, \mu_2)\)-ergodic, if \(Z \in \text{SBC} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\) and satisfies

\[
\lim_{m \to +\infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E}\|Z(t)\|^p d\mu_1(t) = 0.
\]

The collection of all such stochastic processes is denoted by \(\text{SO} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\).

Proposition 2.6. If \(\mu_1, \mu_2 \in \mathfrak{M}\) satisfy (A1), then \(\text{SO} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2), \| \cdot \|_\infty\) is a Banach space.

Proof. It is easy to see that \(\text{SO} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\) is a vector subspace of the Banach space \(\text{SBC} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\). To complete the proof, we need to prove that the space \(\text{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\) is closed in \(\text{SBC} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\). Let \((Z_n)_n\) be a sequence in \(\text{SO} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\) such that \(\lim_{n \to +\infty} \|Z_n - Z\|_\infty = 0\). Since \(\mu_2(\mathbb{R}) = +\infty\), it follows that \(\mu_2([-m, m]) > 0\) for \(m\) sufficiently large. Then, by using the inequality

\[
\int_{-m}^{m} \mathbb{E}\|Z(t)\|^p d\mu_1(t) \leq 2^{p-1} \int_{-m}^{m} \mathbb{E}\|Z_n(t) - Z(t)\|^p d\mu_1(t) + 2^{p-1} \int_{-m}^{m} \mathbb{E}\|Z_n(t)\|^p d\mu_1(t),
\]

we can conclude
we have 
\[
\frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \|Z(t)\|^p d\mu_1(t) \leq 2^{p-1} \frac{\mu_1([[-m, m]])}{\mu_2([-m, m])} \|Z_n - Z\|_{\infty}^p \\
+ 2^{p-1} \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \|Z_n(t)\|^p d\mu_1(t).
\]
Thus, in view of (A1), and the fact that \((Z_n)_n \in SO(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\), we obtain
\[
\limsup_{m \to +\infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \|Z(t)\|^p d\mu_1(t) \leq 2^{p-1} \text{cst.} \|Z_n - Z\|_{\infty}^p, \text{ for all } n \in \mathbb{N}.
\]
Finally, since \(\lim_{n \to +\infty} \|Z_n - Z\|_{\infty}^p = 0\), we conclude that
\[
\lim_{m \to +\infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \|Z(t)\|^p d\mu_1(t) = 0.
\]
The proof is complete. □

**Proposition 2.7** ([4]). Let \(\mu_1, \mu_2 \in \mathfrak{N}\), \(J\) be a bounded interval (eventually \(J = \emptyset\)). Moreover, suppose that (A1) hold and \(Z \in SBC(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\). Then the following assertions are equivalent:

1. \(Z \in SO(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\).
2. \(\lim_{m \to +\infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} J \mathbb{E} \|Z(t)\|^p d\mu_1(t) = 0\).
3. For any \(\upsilon > 0\), \(\lim_{m \to +\infty} \frac{\mu_1(t \in [-m, m] \setminus J \mathbb{E} \|Z(t)\|^p > \upsilon)}{\mu_2(t \in [-m, m])} = 0\).

**Definition 2.8.** Let \(\mu_1, \mu_2 \in \mathfrak{N}\). A continuous stochastic processes \(Z : \mathbb{R} \to L^p(\mathcal{P}, \mathcal{E})\) is said to be \(\mu_1, \mu_2\)-pseudo almost periodic (p-mean (\(\mu_1, \mu_2\))-s.p.a.p. for short) if it can be expressed as \(Z = Z^a + Z^c\), where \(Z^a \in SAP(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\) and \(Z^c \in SO(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\). We denote the collection of all such stochastic processes by \(SAP(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\)

**Theorem 2.9** ([4]). Let \(\mu_1, \mu_2 \in \mathfrak{N}\) satisfy (A2), then \(\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\) is invariant by translation.

**Theorem 2.10** ([4]). Let \(\mu_1, \mu_2 \in \mathfrak{N}\) and \(Z \in SAP(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\) be such that \(Z = Z^a + Z^c\), where \(Z^a \in SAP(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\) and \(Z^c \in SO(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\). If \(\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\) is invariant by translation, then
\[
\{Z^a(t) : t \in \mathbb{R}\} \subset \{Z(t) : t \in \mathbb{R}\}.
\]

**Theorem 2.11.** Let \(\mu_1, \mu_2 \in \mathfrak{N}\) satisfy (A2). Then the decomposition of a p-mean \((\mu_1, \mu_2)\)-s.p.a.p. stochastic processes in the form \(Z = Z^a + Z^c\), where \(Z^a \in SAP(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\) and \(Z^c \in SO(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\) is unique.

**Proof.** Suppose that \(Z = Z_1^a + Z_2^a = Z_3^a + Z_4^a\), where \(Z_1^a, Z_2^a \in SAP(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\) and \(Z_3^c, Z_2^c \in SO(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\), then
\[
0 = (Z_1^a - Z_2^a) + (Z_3^a - Z_2^a) \in SAP(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2),
\]
where \(Z_1^a - Z_2^a \in SAP(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\) and \(Z_3^a - Z_2^a \in SO(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mu_2)\). Then from Theorem 2.10, we obtain \((Z_1^a - Z_2^a)(\mathbb{R}) \subset \{0\}\). Consequently, \(Z_1^a = Z_2^a\) and \(Z_3^a = Z_4^a\). □

**Remark 2.12.** \(Z^a\) and \(Z^c\) in Definition 2.8 are called the p-mean almost periodic component and the p-mean \((\mu_1, \mu_2)\)-ergodic perturbation of the stochastic processes \(Z\) respectively.
Theorem 2.13 (\[\text{[H]}\]). If $\mu_1, \mu_2 \in \mathfrak{H}$ satisfy (A1) and (A2), then
\[ (\mathcal{S}\mathcal{A}\mathcal{P}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})), \| \cdot \|_\infty) \]
is a Banach space.

Let $(\mathcal{E}_1, \| \cdot \|_1)$, $(\mathcal{E}_2, \| \cdot \|_2)$ be Banach spaces, and $L^p(\mathcal{P}, \mathcal{E}_1)$, $L^p(\mathcal{P}, \mathcal{E}_2)$ be corresponding $L^p$-spaces. Consider the following spaces of stochastic processes
\[
\begin{align*}
\mathcal{S}\mathcal{A}\mathcal{P}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2)) \\
= \{ F(\cdot, Y) \in \mathcal{S}\mathcal{A}\mathcal{P}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2)) \text{ for any } Y \in L^p(\mathcal{P}, \mathcal{E}_1) \}, \\
\mathcal{S}\mathcal{O}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2}) \\
= \{ F(\cdot, Y) \in \mathcal{S}\mathcal{O}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2}) \text{ for any } Y \in L^p(\mathcal{P}, \mathcal{E}_1) \}.
\end{align*}
\]

Definition 2.14. Let $\mu_1, \mu_2 \in \mathfrak{H}$. A stochastically continuous processes $F : \mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1) \to L^p(\mathcal{P}, \mathcal{E}_2)$ is said to be p-mean $(\mu_1, \mu_2)$-pseudo almost periodic in $t \in \mathbb{R}$ for any $Y \in L^p(\mathcal{P}, \mathcal{E}_1)$, if it can be expressed as $F = F^a + F^e$, where $F^a \in \mathcal{S}\mathcal{A}\mathcal{P}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2))$ and $F^e \in \mathcal{S}\mathcal{O}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$. We denote the collection of all such stochastically continuous processes by $\mathcal{S}\mathcal{P}\mathcal{A}\mathcal{P}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$.

Theorem 2.15. Let $\mu_1, \mu_2 \in \mathfrak{H}$ satisfy (A2). Suppose that $F \in \mathcal{S}\mathcal{P}\mathcal{A}\mathcal{P}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$ satisfies Lipschits condition in the second variable, that is, there exists $L > 0$ such that for any $Y_1, Y_2 \in L^p(\mathcal{P}, \mathcal{E}_1)$ and for all $t \in \mathbb{R}$
\[ E\| F(t, Y_1) - F(t, Y_2) \|^2_p \leq L E\| Y_1 - Y_2 \|^2_p. \tag{2.1} \]

Then $F(\cdot, Z(\cdot)) \in \mathcal{S}\mathcal{P}\mathcal{A}\mathcal{P}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$ for any $Z \in \mathcal{P}\mathcal{A}\mathcal{P}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_1), \mu_{1,2})$.

Proof. From Definitions 2.8 and 2.14 let $F = F^a + F^e$ and $Z = Z^a + Z^e$, where $F^a \in \mathcal{S}\mathcal{A}\mathcal{P}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2))$, $F^e \in \mathcal{S}\mathcal{O}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$, $Z^a \in \mathcal{S}\mathcal{A}\mathcal{P}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_1), \mu_{1,2})$, and $Z^e \in \mathcal{S}\mathcal{O}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_1), \mu_{1,2})$. Then, we can write
\[
\begin{align*}
F(t, Z(t)) &= F^a(t, Z^a(t)) + [F(t, Z(t)) - F(t, Z^a(t))] + [F(t, Z^a(t)) - F^a(t, Z^a(t))] \\
&= F^a(t, Z^a(t)) + [F(t, Z(t)) - F(t, Z^a(t))] + F^e(t, Z^a(t)).
\end{align*}
\]

First, we claim that $F^a(\cdot, Z^e(\cdot)) \in \mathcal{S}\mathcal{A}\mathcal{P}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2))$. In fact, since $F \in \mathcal{S}\mathcal{P}\mathcal{A}\mathcal{P}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$, then for all $Y \in L^p(\mathcal{P}, \mathcal{E}_1)$, we have
\[
F(\cdot, Y) \in \mathcal{S}\mathcal{P}\mathcal{A}\mathcal{P}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2}).
\]

Hence, we can write $F(\cdot, Y) = F^a(\cdot, Y) + F^e(\cdot, Y)$ with $F^a(\cdot, Y) \in \mathcal{S}\mathcal{A}\mathcal{P}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2))$ and $F^e(\cdot, Y) \in \mathcal{S}\mathcal{O}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$. Since $\| F^a(\cdot, Y_1) - F^a(\cdot, Y_2) \|_2$ is almost periodic component of the p-mean $(\mu_1, \mu_2)$-s.p.a.p. function $\| F(\cdot, Y_1) - F(\cdot, Y_2) \|_2$, by using Theorem 2.10, we deduce that $\| F^a(\cdot, Y_1) - F^a(\cdot, Y_2) \|_\infty \leq \| F(\cdot, Y_1) - F(\cdot, Y_2) \|_\infty$, which implies that, for any $t \in \mathbb{R}$ and $Y_1, Y_2 \in L^p(\mathcal{P}, \mathcal{E}_1)$
\[ E\| F^a(t, Y_1) - F^a(t, Y_2) \|^2_p \leq E\| F(t, Y_1) - F(t, Y_2) \|^2_p \leq L E\| Y_1 - Y_2 \|^2_p. \]

Let us define $K = \{ Z^a(t) : t \in \mathbb{R} \}$. Since $Z^a$ belongs to $\mathcal{S}\mathcal{A}\mathcal{P}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_1))$, it follows that $K$ is a compact set. Therefore, using \[\text{[H]}\] Theorem 4.4, we deduce that $G(\cdot, Z^a(\cdot)) \in \mathcal{S}\mathcal{A}\mathcal{P}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2))$. 


Next, we claim that $F(\cdot, Z(\cdot)) - F(\cdot, Z^a(\cdot)) \in SO(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$. By using the Lipschitz condition, we obtain

$$
\lim_{m \to \infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^m \mathbb{E}[F(t, Z(t)) - F(t, Z^a(t))]_2^p d\mu_1(t)
\leq \lim_{m \to \infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^m L.\mathbb{E}[Z(t) - Z^a(t)]_2^p d\mu_1(t)
\leq \lim_{m \to \infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^m L.\mathbb{E}[Z^a(t)]_2^p d\mu_1(t) = 0.
$$

Finally, it remains to show that $F(\cdot, Z^a(\cdot)) \in SO(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$. Indeed, in the view of \([2.1]\) and Theorem \([2.10]\), it follows that

$$
\mathbb{E}[F^a(t, Y_1) - F^a(t, Y_2)]_2^p
= \mathbb{E}[F(t, Y_1) - F^a(t, Y_1) - F(t, Y_2) + F^a(t, Y_2)]_2^p
\leq 2^{p-1} \mathbb{E}[F(t, Y_1) - F(t, Y_2)]_2^p + 2^{p-1} \mathbb{E}[F^a(t, Y_1) - F^a(t, Y_2)]_2^p
\leq 2^p \mathbb{E}[|Y_1 - Y_2|]_2^p.
$$

Since $\mathbb{K} = \{Z^a(t) : t \in \mathbb{R}\}$ is compact, for $\epsilon > 0$, there exist $Y_1, \ldots, Y_k \in \mathbb{K}$, such that

$$
\mathbb{K} \subset \bigcup_{i=1}^k B(Y_i, \frac{\epsilon}{2^{2p-1} \omega L}),
$$

where

$$
\omega := \limsup_{m \to +\infty} \frac{\mu_1([-m, m])}{\mu_2([-m, m])} < +\infty,
$$

$$
B(Y_i, \frac{\epsilon}{2^{2p-1} \omega L}) := \left\{ Y \in \mathbb{K} : \mathbb{E}[|Y - Y_i|]_2^p \leq \frac{\epsilon}{2^{2p-1} \omega L} \right\}.
$$

By using \([2.2]\) along with the above result, we obtain

$$
\mathbb{K} \subset \bigcup_{i=1}^k \left\{ Y \in \mathbb{K} : \forall t \in \mathbb{R}, \mathbb{E}[F^a(t, Y) - F^a(t, Y_i)]_2^p \leq \frac{\epsilon}{2^{2p-1} \omega L} \right\}.
$$

Let $t \in \mathbb{R}$ and $Y \in \mathbb{K}$, then there exists $i_* \in \{1, \ldots, k\}$ such that

$$
\mathbb{E}[|F^a(t, Y) - F^a(t, Y_{i_*})|]_2^p \leq \frac{\epsilon}{2^{2p-1} \omega}.
$$

Therefore,

$$
\mathbb{E}[F^a(t, Z^a(t))]_2^p \leq 2^{p-1} \mathbb{E}[|F^a(t, Z^a(t)) - F^a(t, Y_{i_*})|]_2^p + 2^{p-1} \mathbb{E}[|F^a(t, Y_{i_*})|]_2^p
\leq \frac{\epsilon}{\omega} + 2^{p-1} \mathbb{E}[|F^a(t, Y_{i_*})|]_2^p
\leq \frac{\epsilon}{\omega} + 2^{p-1} \sum_{i=1}^k \mathbb{E}[|F^a(t, Y_i)|]_2^p.
$$

Since for all $i \in \{1, \ldots, k\}$,

$$
\lim_{m \to \infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^m \mathbb{E}[F^a(t, Y_i)]_2^p d\mu_1(t) = 0,
$$

by using (A1) and \([2.3]\), we obtain

$$
\limsup_{m \to \infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^m \mathbb{E}[|F^a(t, Z^a(t))|]_2^p d\mu_1(t) \leq \epsilon, \text{ for all } \epsilon > 0.
$$
which further implies
\[
\lim_{m \to \infty} \frac{1}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|F^t(t,Z^a(t))\|^2 dt = 0.
\]
Therefore, \(H(\cdot, Z^a(\cdot)) \in SO(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})\). The proof is complete. \(\square\)

3. Application to integro-differential stochastic evolution equations

In this section, we establish some sufficient conditions to ensure the existence, uniqueness and stability of \(p\)-mean (s.p.a.p.) mild solution of (1.1).

**Definition 3.1.** An \(\mathcal{F}_t\) progressively measurable processes \((Z(t))_{t \in \mathbb{R}}\) is called mild solution of (1.1) if it satisfies the following stochastic integral equation

\[
Z(t) = U(t,a)Z(a) + \int_a^t U(t,s)F_1(s,Z(s))ds + \int_a^t U(t,s)\int_a^s Q(s-\zeta)F_2(\zeta,Z(\zeta))d\zeta ds + \int_a^t U(t,s)\int_a^s R(s-\zeta)G(\zeta,Z(\zeta))d\mathcal{W}(\zeta)ds
\]

for all \(t \geq a\) and each \(a \in \mathbb{R}\).

The Acquistapace-Terreni conditions (ATC, for short), which was firstly introduced in \[1\], play an important role in the study of non-autonomous evolution equations. We state it below for the readers’ convenience.

**Definition 3.2.** A family of closed linear operators \(A(t)\) for \(t \in \mathbb{R}\) on a Banach space \((\mathcal{E}, \| \cdot \|)\) with domain \(D(A(t))\) (possibly not densely defined) satisfies ATC, if there exist constants \(w > 0\), \(\gamma \in (\frac{\pi}{2}, \pi)\), \(K_1, K_2 \geq 0\) and \(\nu_1, \nu_2 \in (0,1]\) with \(\nu_1 + \nu_2 > 1\) such that

\[
S_\gamma \cup \{0\} \subset \rho(A(t)-w),
\]

\[
\|R(\lambda, A(t)-w)\| \leq \frac{K_2}{1+|\lambda|},
\]

\[
\|(A(t)-w)R(\lambda, A(t)-w)[R(w, A(t)) - R(w, A(s))]| \| \leq K_2 |t-s|^{\nu_1} |\lambda|^{-\nu_2},
\]

for all \(t, s \in \mathbb{R}, \lambda \in S_\gamma := \{\lambda \in \mathbb{C} - \{0\} : |\arg \lambda| \leq \gamma\} \).

**Lemma 3.3.** Let \(A(t)\) be a family of closed linear operators which satisfies ATC. Then there exists a unique evolution family \(\{U(t,s)\}_{-\infty < s \leq t < +\infty}\) on \(L^p(\mathcal{P}, \mathcal{E})\), which governs the linear part of Eq (1).

We shall use the following assumptions:
(A3) The family of operators \(A(t)\) on \(L^p(\mathcal{P}, \mathcal{E})\) satisfies ATC, and the evolution family associated with \(A(t)\) is exponentially stable, that is, there exists two numbers \(M, \kappa > 0\) such that

\[
\|U(t,s)\| \leq Me^{-\kappa(t-s)}, \quad \text{for all } t, s \in \mathbb{R}, \text{ such that } t \geq s.
\]

(A4) \(R(w, A(\cdot)) \in AP(\mathbb{R}, \mathcal{L}(L^p(\mathcal{P}, \mathcal{E})))\), for \(w\) in Definition 3.2.
(A5) The processes \( F_i : \mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}) \to L^p(\mathcal{P}, \mathcal{E}) \) \((i = 1, 2)\) and \( G : \mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}) \to L^p(\mathcal{P}, \mathcal{L}^2)\) are p-mean \((\mu_1, \mu_2)\)-pseudo almost periodic in \(t \in \mathbb{R}\) for any \( Y \in L^p(\mathcal{P}, \mathcal{E})\). Moreover, \( F_1, F_2, \) and \( G \) are Lipschitz in the following sense: there exists \( L_i > 0 \) \((i = 1, 2, 3)\) such that
\[
\mathbb{E}\|F_i(t, Y_1) - F_i(t, Y_2)\|^p \leq L_i, \mathbb{E}\|Y_1 - Y_2\|^p, \quad i = 1, 2,
\]
\[
\mathbb{E}\|G(t, Y_1) - G(t, Y_2)\|^p \leq L_3, \mathbb{E}\|Y_1 - Y_2\|^p,
\]
for all stochastic processes \( Y_1, Y_2 \in L^p(\mathcal{P}, \mathcal{E}) \) and \( t \in \mathbb{R} \).

The next Lemma, which can be seen as an immediate consequence of [28, Proposition 4.4] is essential to study the existence of p-mean \((\mu_1, \mu_2)\)-s.p.a.p. mild solutions.

**Lemma 3.4.** Suppose that (A3), (A4) hold. Then, for any \( \epsilon > 0 \) and \( h > 0 \), there exists \( l = l(\epsilon) > 0 \) such that every interval of length \( l \) contains at least a number \( r \)
\[
\|U(t + r, s + r) - U(t, s)\| \leq e^{-\frac{\eta}{2}(t-s)}, \quad \text{for all } t - s \geq h.
\]

**Lemma 3.5.** Let \( \mu_1, \mu_2 \in \mathcal{A} \) satisfy (A1)–(A4) hold. Furthermore, assume that \( Z \in \text{SPAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1.2})\). Then the function
\[
\Gamma : t \mapsto \int_{-\infty}^t U(t, s)Z(s)ds,
\]
belongs to \( \text{SPAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1.2})\).

**Proof.** For \( Z \in \text{SPAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1.2})\), there exist \( Z^a \in \text{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\) and \( Z^e \in \text{S\textcircled{O}}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1.2})\), such that \( Z = Z^a + Z^e\). Consequently, we can write
\[
\Gamma(t) = \Gamma_1(t) + \Gamma_2(t)
\]
\[
= \int_{-\infty}^t U(t, s)Z^a(s)ds + \int_{-\infty}^t U(t, s)Z^e(s)ds.
\]

First, we show that \( \Gamma_1 \in \text{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\). In fact, since \( Z^a \in \text{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))\), so according to Lemma 3.4, for a given \( \epsilon > 0 \), one can find \( l(\epsilon) > 0 \) such that for any interval of length \( l(\epsilon) \) contains at least a number \( r \) with the property that
\[
\|U(t + r, s + r) - U(t, s)\| \leq e^{-\frac{\eta}{2}(t-s)}, \quad \text{for all } t - s \geq \epsilon.
\]
and
\[
\mathbb{E}\|Z^a(t + r) - Z^a(t)\|^p < \eta, \quad \text{for all } t \in \mathbb{R},
\]
where \( \eta = \eta(\epsilon) \to 0 \) as \( \epsilon \to 0 \). By using (3.2), (3.3), Hölder’s inequality, and that
\[
|x + y + z|^p \leq 3^{p-1}(|x|^p + |y|^p + |z|^p),
\]
it follows that
\[
\mathbb{E}\|\Gamma_1(t + r) - \Gamma_1(t)\|^p
\]
\[
= \mathbb{E}\left\|\int_{-\infty}^{t+r} U(t + r, s)Z^a(s)ds - \int_{-\infty}^t U(t, s)Z^a(s)ds\right\|^p
\]
\[
= \mathbb{E}\left\|\int_0^{t+r} U(t + r, t + r - s)Z^a(t + r - s)ds - \int_0^t U(t, t - s)Z^a(t - s)ds\right\|^p
\]
\[
\leq 3^{p-1}\mathbb{E}\left[\int_0^{t+r} \|U(t + r, t + r - s)\| \cdot \|Z^a(t + r - s) - Z^a(t - s)\|ds\right]^p
\]
Next, we check that $\Gamma \leq 3 \epsilon \sup_{\mu} \left\| Z^a(t + r - s) - Z^a(t - s) \right\| ds$

\[
\leq 3^{p-1} M^p \left[ \int_0^1 e^{-\kappa s} \left\| Z^a(t + r - s) - Z^a(t - s) \right\| ds \right]^{p-1} \int_0^1 e^{-\kappa s} \left\| Z^a(t) \right\| ds
\]

\[
+ 3^{p-1} 2^{p-1} M^p \left[ \int_0^1 e^{-\kappa s} \left\| Z^a(t) \right\| ds \right]^{p-1} \int_0^1 e^{-\kappa s} \left\| Z^a(t - s) \right\| ds
\]

\[
+ 3^{p-1} M^p \left[ \int_0^1 e^{-\kappa s} \left\| Z^a(t - s) \right\| ds \right]^{p-1} \int_0^1 e^{-\kappa s} \left\| Z^a(t) \right\| ds
\]

\[
\leq 3^{p-1} M^p \left[ \int_0^1 e^{-\kappa s} \left\| Z^a(t) \right\| ds \right]^{p-1} \int_0^1 e^{-\kappa s} \left\| Z^a(t - s) \right\| ds
\]

\[
+ 3^{p-1} 2^{p-1} M^p \left[ \int_0^1 e^{-\kappa s} \left\| Z^a(t - s) \right\| ds \right]^{p-1} \int_0^1 e^{-\kappa s} \left\| Z^a(t) \right\| ds
\]

\[
+ 3^{p-1} M^p \left[ \int_0^1 e^{-\kappa s} \left\| Z^a(t) \right\| ds \right]^{p-1} \int_0^1 e^{-\kappa s} \left\| Z^a(t - s) \right\| ds
\]

\[
\leq 3^{p-1} M^p (1/\kappa)^p \eta(\epsilon) + 3^{p-1} 2^p M^p \left\| Z \right\|^p \| \epsilon \|_\infty^p + 3^{p-1} (2/\kappa)^p \left\| Z \right\|^p \| \epsilon \|_\infty^p
\]

which implies that $\Gamma_1 \in S\mathcal{AP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))$.

Next, we check that $\Gamma_2 \in S\mathcal{O}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}, \mu_1, 2))$, that is

\[
\lim_{m \to +\infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^m E|\Gamma_2(t)|^p d\mu_1(t) = 0.
\]

Let us denote $A := \{ s : s \leq t \}$ and $B := \{ v : v \geq 0 \}$. Applying Hölder’s inequality and Fubini’s theorem, for $m > 0$, we obtain

\[
\frac{1}{\mu_2([-m, m])} \int_{-m}^m E|\Gamma_2(t)|^p d\mu_1(t)
\]

\[
= \frac{1}{\mu_2([-m, m])} \int_{-m}^m E \left\| \int_{-\infty}^t U(t, s) Z^c(s) ds \right\|^p d\mu_1(t)
\]

\[
\leq \frac{1}{\mu_2([-m, m])} \int_{-m}^m E \left\| \int_{-\infty}^t U(t, s) \right\|^p Z^c(s) ds \left\| d\mu_1(t)
\]

\[
\leq \frac{M^p}{\mu_2([-m, m])} \int_{-m}^m \left\{ \int_{-\infty}^t e^{-\kappa(t-s)} ds \right\}^{p-1} \int_{-\infty}^t e^{-\kappa(t-s)} \left\| Z^c(s) \right\|^p ds \left\| d\mu_1(t)
\]

\[
\leq \frac{M^p}{\kappa^{p-1} \mu_2([-m, m])} \int_{-m}^m \left\{ \int_{-\infty}^t e^{-\kappa(t-s)} \left\| Z^c(s) \right\|^{p \chi_A} ds \right\} d\mu_1(t)
\]

\[
\leq \frac{M^p}{\kappa^{p-1} \mu_2([-m, m])} \int_{-m}^m \left\{ \int_{-\infty}^t e^{-\kappa(t-s)} \left\| Z^c(s) \right\|^{p \chi_A} d\mu_1(t) \right\} ds.
\]
By making change of variables \( v = t - s \), it follows that
\[
\frac{1}{\mu_2([-m, m])} \int_{-m}^{m} E\|\Gamma_2(t)\|^p d\mu_1(t)
\]
\[
\leq \frac{M^p}{\kappa^{p-1} \mu_2([-m, m])} \int_{-\infty}^{\infty} \left\{ \int_{-m}^{m} e^{-\kappa v} E\|Z^c(t-v)\|^p d\mu_1(t) \right\} (-dv)
\]
\[
\leq \frac{M^p}{\kappa^{p-1} \mu_2([-m, m])} \int_{-\infty}^{\infty} \left\{ \int_{-m}^{m} e^{-\kappa v} E\|Z^c(t-v)\|^p d\mu_1(t) \right\} dv
\]
\[
\leq \frac{M^p}{\kappa^{p-1} \mu_2([-m, m])} \int_{0}^{+\infty} \left\{ \frac{e^{-\kappa v}}{\mu_2([-m, m])} \int_{-m}^{m} E\|Z^c(t-v)\|^p d\mu_1(t) \right\} dv.
\]
One can see that
\[
\left| \frac{e^{-\kappa v}}{\mu_2([-m, m])} \int_{-m}^{m} E\|Z^c(t-v)\|^p d\mu_1(t) \right| \leq e^{-\kappa v} \|Z^c\|^p \frac{\mu_1([-m, m])}{\mu_2([-m, m])}
\]
for all \( v \geq 0 \). Since \( \mu_1 \) and \( \mu_2 \) satisfy (A2), from Theorem 2.9 we have
\[
|t \mapsto X^c(t-v)| \in \text{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}).
\]
Then, by using (A2) and the Lebesgue dominate convergence theorem, we obtain
\[
\lim_{m \to +\infty} \frac{M^p}{\kappa^{p-1}} \int_{0}^{+\infty} \left\{ \frac{e^{-\kappa v}}{\mu_2([-m, m])} \int_{-m}^{m} E\|Z^c(t-v)\|^p d\mu_1(t) \right\} dv = 0,
\]
which implies that
\[
\lim_{m \to +\infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} E\|\Gamma_2(t)\|^p d\mu_1(t).
\]
Finally, we obtain \( \Gamma = \Gamma_1 + \Gamma_2 \in \text{SPAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \).

**Lemma 3.6.** Let \( \mu_1, \mu_2 \in \mathfrak{M} \) satisfy (A1), (A2) and \( Z \in \text{SPAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \).
Then the function
\[
\Lambda : t \mapsto \int_{-\infty}^{t} Q(t-\zeta)Z(\zeta)d\zeta,
\]
belongs to \( \text{SPAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \).

**Proof.** For \( Z \in \text{SPAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \), there exist \( Z^a, Z^c \in \text{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})) \) and \( Z^c \in \text{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \), such that \( Z = Z^a + Z^c \). Hence, we can write
\[
\Lambda(t) = \Lambda_1(t) + \Lambda_2(t)
\]
\[
= \int_{-\infty}^{t} Q(t-\zeta)Z^a(\zeta)d\zeta + \int_{-\infty}^{t} Q(t-\zeta)Z^c(\zeta)d\zeta
\]
\[
= \int_{0}^{+\infty} Q(\zeta)Z^a(t-\zeta)d\zeta + \int_{0}^{+\infty} Q(\zeta)Z^c(t-\zeta)d\zeta.
\]
First, let us show that \( \Lambda_1 \in \text{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})) \). Since \( Z^a \in \text{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})) \), so for any given \( \epsilon > 0 \), one can find \( l(\epsilon) > 0 \) such that for any interval of length \( l(\epsilon) \) contains at least a number \( r \) such that
\[
E\|Z^a(t+r) - Z^a(t)\|^p < \frac{\epsilon}{\|Q\|_{L^p(0, +\infty)}}, \quad \text{for all } t \in \mathbb{R}.
\]
Now by using Hölder’s inequality, we obtain
\[
E\|\Lambda_1(t+r) - \Lambda_1(t)\|^p
\]
\[ M. AYACHI, S. ABBAS \]

SO for all which implies that

\[ \text{Finally, we obtain } \Lambda = \Lambda \]

Let Lemma 3.7.

\[ \text{which implies that } \Lambda \]

Then the function

\[ \text{belongs to } \]

Next, we check that \( \Lambda_2 \in \mathcal{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})) \); that is

\[ \lim_{m \to +\infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \| \Lambda_2(t) \|^p d\mu_1(t) = 0. \]

By using Hölder’s inequality and Fubini’s theorem, for \( m > 0 \), we obtain

\[
\begin{align*}
\frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \| \Lambda_2(t) \|^p d\mu_1(t) \\
= \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \left( \int_{0}^{+\infty} Q(\zeta) Z^e(t - \zeta) d\zeta \right)^p d\mu_1(t) \\
\leq \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \left( \int_{0}^{+\infty} \| Q(\zeta) \| \| Z^e(t - \zeta) \| d\zeta \right)^p d\mu_1(t) \\
\leq \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \left\{ \left( \int_{0}^{+\infty} \| Q(\zeta) \| d\zeta \right)^{p-1} \int_{0}^{+\infty} \| Q(\zeta) \| \mathbb{E} \| Z^e(t - \zeta) \| d\zeta \right\} d\mu_1(t) \\
\leq \| Q \|_{L^1(0, +\infty)}^{p-1} \int_{0}^{+\infty} \left\{ \frac{\| Q(\zeta) \|}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \| Z^e(t - \zeta) \|^p d\mu_1(t) \right\} d\zeta.
\end{align*}
\]

Since

\[
\left| \frac{\| Q(\zeta) \|}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \| Z^e(t - \zeta) \|^p d\mu_1(t) \right| \leq \| Q(\zeta) \| \| Z^e \|^p_{L^\infty} \frac{\mu_1([-m, m])}{\mu_2([-m, m])},
\]

for all \( \zeta \geq 0 \), by using the Lebesgue dominate convergence theorem, (A1), and that the space \( \mathcal{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \) is invariant by translation, we obtain

\[
\lim_{m \to +\infty} \int_{0}^{+\infty} \left\{ \frac{\| Q(\zeta) \|}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \| Z^e(t - \zeta) \|^p d\mu_1(t) \right\} d\zeta = 0,
\]

which implies that

\[
\lim_{m \to +\infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E} \| \Lambda_2(t) \|^p d\mu_1(t).
\]

Finally, we obtain \( \Lambda = \Lambda_1 + \Lambda_2 \in \mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \). \( \square \)

**Lemma 3.7.** Let \( \mu_1, \mu_2 \in \mathfrak{M} \) satisfy (A1), (A2) and \( Z \in \mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, L^0_2), \mu_{1,2}) \).

Then the function

\[ \Delta : t \mapsto \int_{-\infty}^{t} R(t - \zeta) Z(\zeta) d\mathcal{W}(\zeta), \]

belongs to \( \mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \).
Proof. Let us assume that \( Z \in \mathcal{SPA} (\mathbb{R}, L^p(\mathcal{P}, \mathbb{L}_2^0), \mu_{1,2}) \). Then, there exist \( Z^a \in \mathcal{SPA}(\mathbb{R}, L^p(\mathcal{P}, \mathbb{L}_2^0)) \) and \( Z^c \in \mathcal{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathbb{L}_2^0), \mu_{1,2}) \), such that \( Z = Z^a + Z^c \). Hence, we can write

\[
\Delta(t) = \Delta_1(t) + \Delta_2(t) = \int_{-\infty}^{t} R(t - \zeta)Z^a(\zeta)dW(\zeta) + \int_{-\infty}^{t} R(t - \zeta)Z^c(\zeta)dW(\zeta).
\]

First, let us show that \( \Delta_1 \in \mathcal{SPA} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})) \). Since \( Z^a \in \mathcal{SPA} (\mathbb{R}, L^p(\mathcal{P}, \mathbb{L}_2^0)) \), for given \( \epsilon > 0 \), one can find \( l(\epsilon) > 0 \) such that for any interval of length \( l(\epsilon) \) contains at least \( r \) with the property that

\[
E[|Z^a(t+r) - Z^a(t)|^{p/2}] < \frac{\epsilon}{C_p \|R\|_{L^2(0,\infty)}}, \quad \text{for all } t \in \mathbb{R}.
\]

Let \( \overline{W}(\zeta) = W(\zeta + r) - W(r) \) for each \( \zeta \in \mathbb{R} \). Then, \( \overline{W} \) is also a Brownian motion and has the same distribution as \( W \). By making change of variable \( v = \zeta - r \), and using Lemma 2.1, we obtain

\[
E[|\Delta_1(t + r) - \Delta_1(t)|^p] = E\left[ \left| \int_{-\infty}^{t+r} R(t + r - \zeta)X^a(\zeta)d\overline{W}(\zeta) - \int_{-\infty}^{t} R(t - \zeta)Z^a(\zeta)dW(\zeta) \right|^p \right]
\]

\[
= E\left[ \left| \int_{-\infty}^{t} R(t - v)X^a(v + r) d\overline{W}(v) - \int_{-\infty}^{t} R(t - v)Z^a(v)d\overline{W}(v) \right|^p \right]
\]

\[
= E\left[ \left| \int_{-\infty}^{t} R(t - v)[Z^a(v + r) - Z^a(v)] d\overline{W}(v) \right|^p \right]
\]

\[
\leq C_p E\left[ \left( \int_{-\infty}^{t} \|R(t - v)\|^p \|Z^a(v + r) - Z^a(v)\|_{L_2^0}^{2p/2} dv \right)^{p/2} \right]
\]

\[
\leq C_p E\left[ \left( \int_{-\infty}^{t} \|R(t - v)\|^p \|Z^a(v + r) - Z^a(v)\|_{L_2^0}^{2p/2} dv \right)^{p/2} \right],
\]

where \( q > 0 \) solves \( \frac{1}{p/2} + \frac{1}{q/2} = \frac{2}{p} + \frac{2}{q} = 1 \). Then, from Hölder’s inequality, we obtain

\[
E[|\Delta_1(t + r) - \Delta_1(t)|^p] \leq C_p \left[ \int_{-\infty}^{t} \|R(t - v)\|^p dv \right]^{p/2} \int_{-\infty}^{t} \|R(t - v)\|^{2p/2} \|Z^a(v + r) - Z^a(v)\|_{L_2^0}^{p/2} dv
\]

\[
\leq C_p \|R\|_{L^p(0,\infty)}^{p} \sup_{v \in \mathbb{R}} E[|Z^a(v + r) - Z^a(v)|_{L_2^0}^{p}] \leq \epsilon.
\]

Which implies that \( \Delta_1 \in \mathcal{SPA} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})) \).

Next, we check that \( \Delta_2 \in \mathcal{SO} (\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \). Let us denote \( A := \{ \zeta \in \mathbb{R} : \zeta \leq t \} \) and \( B := \{ v : v \geq 0 \} \). By using Hölder’s inequality and Fubini’s theorem, we obtain, for \( m > 0 \):

\[
\frac{1}{\mu_2([-m, m])} \int_{-m}^{m} E[|\Delta_2(t)|^p] d\mu_1(t)
\]

\[
= \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} E \left[ \left| \int_{-\infty}^{t} R(t - \zeta)Z^c(\zeta)dW(\zeta) \right|^p \right] d\mu_1(t)
\]
14 M. AYACHI, S. ABBAS EJDE-2024/24

SO

\[ \Delta = \Delta \]

Finally, we have

\[ \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E}[\Delta_1(t)] d\mu_1(t) \]

By making change of variables \( v = t - \zeta \), we obtain

\[ \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E}[\Delta_2(t)] d\mu_1(t) \]

Since, for all \( v \geq 0 \)

\[ \left| \frac{\|R(v)\|}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E}[X^c(t - v)] d\mu_1(t) \right| \leq \|R(v)\| \mathbb{E}[\mu_2([-m, m])] \]

by using the Lebesgue dominate convergence theorem, (A1), and that the space \( \mathcal{S}_0(\mathbb{R}, L^p(\mathcal{P}, L^2_0), \mu_1, \mathcal{E}) \) is invariant by translation, we obtain

\[ \lim_{m \to +\infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E}[\Delta_2(t)] d\mu_1(t) = 0, \]

which implies that

\[ \lim_{m \to +\infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E}[\Delta_2(t)] d\mu_1(t) = 0. \]

Finally, we have

\[ \Delta = \Delta_1 + \Delta_2 \in SPAP(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_1, \mathcal{F}). \]

**Theorem 3.8.** Let \( \mu_1, \mu_2 \in \mathfrak{M} \) satisfy (A1), (A2). Suppose that (A3)–(A5) hold. Then \( [1,1] \) has a unique p-mean \( (\mu_1, \mu_2) \)-s.p.a.p. mild solution, which can be explicitly expressed as

\[ Z(t) = \int_{-\infty}^{t} U(t, s)F_1(s, Z(s)) ds + \int_{-\infty}^{t} U(t, s) \int_{-\infty}^{s} Q(s - \zeta)F_2(\zeta, Z(\zeta)) d\zeta ds \]

\[ + \int_{-\infty}^{t} U(t, s) \int_{-\infty}^{s} R(s - \zeta)G(\zeta, Z(\zeta))dW(\zeta) ds, \quad t \in \mathbb{R}, \]

whenever

\[ \Theta_p := M^p(1/\kappa)^p \left[ L_1 + L_2\|Q\|_{L^1(0, +\infty)}^p + C_p L_3\|R\|_{L^2(0, +\infty)}^p \right] < (1/3)^{p-1}, \quad (3.4) \]

for \( p > 2 \), and

\[ \Theta_2 := M^2(1/\kappa)^2 \left[ L_1 + L_2\|Q\|_{L^2(0, +\infty)}^2 + L_3\|R\|_{L^2(0, +\infty)}^2 \right] < 1/3, \quad (3.5) \]
Proof. First of all, it is not difficult to see that the stochastic processes
\[
Z(t) = \int_{-\infty}^{t} U(t, s) F_1(s, Z(s)) ds + \int_{-\infty}^{t} U(t, s) \int_{-\infty}^{s} Q(s - \zeta) F_2(\zeta, Z(\zeta)) d\zeta ds
+ \int_{-\infty}^{t} U(t, s) \int_{-\infty}^{s} R(s - \zeta) G(\zeta, Z(\zeta)) d\zeta ds
\]
for \( p = 2 \).

We prove that \( \Gamma \) is a strict contraction mapping on \( SPAP(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \). In fact, for each \( t \in \mathbb{R} \), and \( Z_1, Z_2 \in SPAP(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \), we have

\[
\mathbb{E}\| (\Gamma Z_1)(t) - (\Gamma Z_2)(t) \|^p \leq 3^{p-1} \mathbb{E}\| \int_{-\infty}^{t} U(t, s) [F_1(s, Z_1(s)) - F_1(s, Z_2(s))] ds \|^p
+ 3^{p-1} \mathbb{E}\| \int_{-\infty}^{t} U(t, s) \int_{-\infty}^{s} Q(s - \zeta) [F_2(\zeta, Z_1(\zeta)) - F_2(\zeta, Z_2(\zeta))] d\zeta ds \|^p
+ 3^{p-1} \mathbb{E}\| \int_{-\infty}^{t} U(t, s) \int_{-\infty}^{s} R(s - \zeta) [G(\zeta, Z_1(\zeta)) - G(\zeta, Z_2(\zeta))] d\zeta ds \|^p
\]

Now, we evaluate the first term of the right-hand side with the help of Hölder’s inequality as follows

\[
3^{p-1} \mathbb{E}\| \int_{-\infty}^{t} U(t, s) [F_1(s, Z_1(s)) - F_1(s, Z_2(s))] ds \|^p
\]

\[
\leq 3^{p-1} M^p \mathbb{E}\| \int_{-\infty}^{t} e^{-\kappa(t-s)} [F_1(s, Z_1(s)) - F_1(s, Z_2(s))] ds \|^p
\]
For the third term, we use Hölder's inequality and Lemma 2.1 to obtain
\[
16 M. AYACHI, S. ABBAS EJDE-2024/24
\]
Furthermore, by using the Hölder’s inequality for the second term, we have
\[
3^p \leq 3^p \left[ \int_{-\infty}^{t} e^{-\kappa(t-s)} ds \right]^{p-1} \int_{-\infty}^{t} e^{-\kappa(t-s)} \mathbb{E} ||F_1(s, Z_1(s)) - F_1(s, Z_2(s))||^p ds
\]
\[
\leq 3^p \left[ \int_{-\infty}^{t} e^{-\kappa(t-s)} ds \right]^{p-1} \int_{-\infty}^{t} e^{-\kappa(t-s)} \mathbb{E} ||Z_1(s) - Z_2(s)||^p ds
\]
\[
\leq 3^p \left[ \int_{-\infty}^{t} e^{-\kappa(t-s)} ds \right]^{p-1} \left[ \int_{-\infty}^{t} e^{-\kappa(t-s)} ds \right] \sup_{s \in \mathbb{R}} \mathbb{E} ||Z_1(s) - Z_2(s)||^p
\]
\[
\leq 3^p \left[ \int_{-\infty}^{t} e^{-\kappa(t-s)} ds \right]^{p-1} \left[ \int_{-\infty}^{t} e^{-\kappa(t-s)} ds \right] \sup_{s \in \mathbb{R}} \mathbb{E} ||Z_1(s) - Z_2(s)||^p
\]

For the third term, we use Hölder’s inequality and Lemma 2.1 to obtain
\[
3^p \mathbb{E} \left[ \int_{-\infty}^{t} ||U(t, s)|| \int_{-\infty}^{s} R(s - \zeta)(G(\zeta, Z_1(\zeta))
\]
\[
- G(\zeta, Z_2(\zeta)) d\mathcal{W}(\zeta) ||ds \right]^p
\]
\[
\leq 3^p \mathbb{E} \left[ \int_{-\infty}^{t} e^{-\kappa(t-s)} || \int_{-\infty}^{s} R(s - \zeta)(G(\zeta, Z_1(\zeta)) - G(\zeta, Z_2(\zeta))) d\mathcal{W}(\zeta) ||ds \right]^p
\]
\[
\leq 3^p \mathbb{E} \left[ \int_{-\infty}^{t} e^{-\kappa(t-s)} || \int_{-\infty}^{s} R(s - \zeta)(G(\zeta, Z_1(\zeta)) - G(\zeta, Z_2(\zeta))) d\mathcal{W}(\zeta) \right]^{p-1} \left[ \int_{-\infty}^{t} e^{-\kappa(t-s)} \mathbb{E} \left[ \int_{-\infty}^{s} ||R(s - \zeta)||^2 ||G(\zeta, Z_1(\zeta)|| ds \right]^p
\]
\[
\leq 3^p \mathbb{E} \left[ \int_{-\infty}^{t} e^{-\kappa(t-s)} \mathbb{E} \left[ \int_{-\infty}^{s} ||R(s - \zeta)||^2 ||G(\zeta, Z_1(\zeta)|| ds \right]^p
\]
Hence, 

\[-G(\zeta, Z_2(\zeta))\int_{-2}^{\infty} d\zeta \] ^{p/2} \] 

\[\leq 3^{p-1} M^p C_p (1/\kappa)^{p-1} \int_{-\infty}^{t} e^{-\kappa(t-s)} \left[ \int_{-\infty}^{s} \|R(s-\zeta)\|^{2} \] ^{p/2} \] \[\times E\|G(\zeta, Z_1(\zeta)) - G(\zeta, Z_2(\zeta))\|^{p} d\zeta ds \]

\[\leq 3^{p-1} M^p C_p L_3 (1/\kappa)^{p-1} \int_{-\infty}^{t} e^{-\kappa(t-s)} \left[ \int_{-\infty}^{s} \|R(s-\zeta)\|^{2} \] ^{p/2} \] \[\times E\|Z_1(\zeta) - Z_2(\zeta)\|^{p} d\zeta ds \]

\[\leq 3^{p-1} M^p C_p L_3 (1/\kappa)^{p} \|R\|_{L^2(0, +\infty)}^{p} \|Z_1 - Z_2\|_{\infty}^{p}. \]

Therefore, for each \( t \in \mathbb{R} \), we can deduce that

\[ E\| (\Gamma Z_1)(t) - (\Gamma Z_2)(t)\|^{p} \]

\[\leq 3^{p-1} M^p (1/\kappa)^{p} \left[ L_1 + L_2 \|Q\|_{L^{1}(0, +\infty)}^{p} + C_p L_3 \|R\|_{L^2(0, +\infty)}^{p} \right] \|Z_1 - Z_2\|_{\infty}^{p}. \]

Hence,

\[\|\Gamma Z_1 - \Gamma Z_2\|_{\infty}^{p} \leq 3^{p-1} \Theta_p \|Z_1 - Z_2\|_{\infty}^{2}. \]

For the case \( p = 2 \), by using the same arguments used to prove [27, Theorem 3.3], we obtain

\[\|\Gamma Z_1 - \Gamma Z_2\|_{\infty}^{2} \leq 3^{p-1} \Theta_2 \|Z_1 - Z_2\|_{\infty}^{2}. \]

Which implies that \( \Gamma \) is a contraction. Hence, by the Banach contraction principle, we can deduce that \( \Gamma \) has a unique fixed point \( Z^* \in \text{SPAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}) \), which correspond to the unique \( \mu_{1,2} \)-s.p.a.p mild solution on \( \mathbb{R} \) of (1.1). This completes the proof.

Finally, we investigate the stability of solution of (1.1) obtained in the previous Theorem. First, we recall the definition of stability.

**Definition 3.9.** The unique \( \mu_{1,2} \)-s.p.a.p mild solution \( Z^*(t) \) of (1.1) is said to be stable in \( \mu_{1,2} \)-sense, if for arbitrary \( \epsilon > 0 \), there exists \( \eta > 0 \) such that

\[ E\|Z(t) - Z^*(t)\|^{p} < \epsilon, \text{ for all } t \geq 0, \]

whenever \( E\|Z(0) - Z^*(0)\|^{p} < \eta \), where \( Z(t) \) stands for a solution of (1.1), with initial value \( Z(0) \).

**Theorem 3.10.** Let \( \mu_{1,2} \in \mathcal{R} \) satisfy (A1), (A2). Suppose that (A3)–(A5) hold. Then (1.1) has a unique \( \mu_{1,2} \)-s.p.a.p mild solution which is stable provided that

\[ \Theta_p := M^p (1/\kappa)^{p} \left[ L_1 + L_2 \|Q\|_{L^{1}(0, +\infty)}^{p} + C_p L_3 \|R\|_{L^2(0, +\infty)}^{p} \right] < (1/4)^{p-1}, \] (3.6)

for \( p > 2 \), and

\[ \Theta_2 := M^2 (1/\kappa)^{2} \left[ L_1 + L_2 \|Q\|_{L^{1}(0, +\infty)}^{2} + L_3 \|R\|_{L^2(0, +\infty)}^{2} \right] < 1/4, \] (3.7)

for \( p = 2 \).
Proof. Note that condition (3.7) (resp. (3.6)) implies condition (3.5) (resp. (3.4)). Then, from Theorem 3.3, we know that (1.1) has a unique p-mean $(\mu_1, \mu_2)$-s.p.a.p. mild solution $Z^*$, whose integral form is given by (3.1). Let $Z(t)$ be an arbitrary solution of (1.1) with initial value $Z(0)$. Then

$$
E\|Z(t) - Z^*(t)\|^p = E\|U(t, 0)[Z(0) - Z^*(0)]
$$

$$+
\int_0^t U(t, s) [F_1(s, Z(s)) - F_1(s, Z^*(s))] ds
$$

$$+
\int_0^t U(t, s) \int_0^s Q(s - \zeta)[F_2(\zeta, Z(\zeta)) - F_2(\zeta, Z^*(\zeta))] d\zeta ds
$$

$$+
\int_0^t U(t, s) \int_0^s R(s - \zeta)[G(\zeta, Z(\zeta)) - G(\zeta, Z^*(\zeta))] d\zeta dW(\zeta) ds|^p.
$$

Now assume that $p > 2$. With the help of Hölder’s inequality, for any $t \geq 0$, we have

$$
E\|Z(t) - Z^*(t)\|^p
\leq
4p^{-1}E\|U(t, 0)[Z(0) - Z^*(0)]\|^p
$$

$$+
4p^{-1}E\left[\int_0^t \|U(t, s)\| |F_1(s, Z(s)) - F_1(s, Z^*(s))| ds\right]^p
$$

$$+
4p^{-1}E\left[\int_0^t \|U(t, s)\| \|Q(s - \zeta)[F_2(\zeta, Z(\zeta)) - F_2(\zeta, Z^*(\zeta))] d\zeta\| ds\right]^p
$$

$$+
4p^{-1}E\left[\int_0^t \|U(t, s)\| \|R(s - \zeta)[G(\zeta, Z(\zeta)) - G(\zeta, Z^*(\zeta))] d\zeta dW(\zeta)\| ds\right]^p
$$

$$\leq
4p^{-1}M^p e^{-\kappa t}E\|Z(0) - Z^*(0)\|^p
$$

$$+
4p^{-1}M^p \left[\int_0^t e^{-\kappa(t-s)}\|F_1(s, Z(s)) - F_1(s, Z^*(s))\| ds\right]^p
$$

$$+
4p^{-1}M^p \left[\int_0^t e^{-\kappa(t-s)}\|Q(s - \zeta)[F_2(\zeta, Z(\zeta)) - F_2(\zeta, Z^*(\zeta))] d\zeta\| ds\right]^p
$$

$$+
4p^{-1}M^p \left[\int_0^t e^{-\kappa(t-s)}\|R(s - \zeta)[G(\zeta, Z(\zeta)) - G(\zeta, Z^*(\zeta))] d\zeta dW(\zeta)\| ds\right]^p
$$

$$\leq
4p^{-1}M^p e^{-\kappa t}E\|Z(0) - Z^*(0)\|^p
$$

$$+
4p^{-1}M^p \left[\int_0^t e^{-\kappa(t-s)} ds\right]^{-p} \int_0^t e^{-\kappa(t-s)} E\|F_1(s, Z(s)) - F_1(s, Z^*(s))\|^p ds
$$

$$+
4p^{-1}M^p \left[\int_0^t e^{-\kappa(t-s)} ds\right]^{-p} \int_0^t e^{-\kappa(t-s)} E\|Q(s - \zeta)(F_2(\zeta, Z(\zeta))) ds
$$

$$+
4p^{-1}M^p \left[\int_0^t e^{-\kappa(t-s)} ds\right]^{-p} \int_0^t e^{-\kappa(t-s)} E\|Q(s - \zeta)(F_2(\zeta, Z(\zeta))) ds
$$
\[-F_2(\zeta, Z^*(\zeta))d\zeta\parallel p ds\]
\[+ 4p^{-1}M^p \left[ \int_0^t e^{-\kappa(t-s)} ds \right]^{p-1} \int_0^t e^{-\kappa(t-s)} \mathbb{E}\| \int_0^s R(s-\zeta)(G(\zeta, Z(\zeta))
- G(\zeta, Z^*(\zeta)))dV(\zeta)\parallel p ds\]
\[\leq 4p^{-1}M^p \mathbb{E}\| Z(0) - Z^*(0) \parallel^p + 4p^{-1}\Theta_p \sup_{t \in \mathbb{R}} \mathbb{E}\| Z(t) - Z^*(t) \parallel^p,\]

which implies that
\[
\sup_{t \in \mathbb{R}} \mathbb{E}\| Z(t) - Z^*(t) \parallel^p \leq 4p^{-1}M^p \mathbb{E}\| Z(0) - Z^*(0) \parallel^p + 4p^{-1}\Theta_p \sup_{t \in \mathbb{R}} \mathbb{E}\| Z(t) - Z^*(t) \parallel^p.
\]
That is
\[
\sup_{t \in \mathbb{R}} \mathbb{E}\| Z(t) - Z^*(t) \parallel^p \leq \frac{4p^{-1}M^p}{1 - 4p^{-1}\Theta_p} \mathbb{E}\| Z(0) - Z^*(0) \parallel^p.
\]
So, for all \( t > 0 \), we obtain
\[
\mathbb{E}\| Z(t) - Z^*(t) \parallel^p \leq \frac{4p^{-1}M^p}{1 - 4p^{-1}\Theta_p} \mathbb{E}\| Z(0) - Z^*(0) \parallel^p.
\]

For \( \epsilon > 0 \), choosing \( 0 < \eta < \epsilon \frac{1 - 4p^{-1}\Theta_p}{4M^2} \), we obtain
\[
\mathbb{E}\| Z(0) - Z^*(0) \parallel^p < \eta \implies \mathbb{E}\| Z(t) - Z^*(t) \parallel^p \leq \epsilon, \text{ for all } t > 0,
\]

According to Definition 3.9, we can conclude that (1.1) has a unique p-mean \((\mu_1, \mu_2)\)-s.p.a.p. mild solution which is stable in p-mean sense.

Now assume that \( p = 2 \). By help of the Cauchy-Schwartz inequality, the Ito’s isometry and the Fubini’s theorem, we have
\[
\mathbb{E}\| Z(t) - Z^*(t) \parallel^2
\leq 4\mathbb{E}\| U(t, 0) [Z(0) - Z^*(0)] \parallel^2
\[+ 4\mathbb{E}\| \int_0^t U(t, s) [F_1(s, Z(s)) - F_1(s, Z^*(s))] ds \parallel^2\]
\[+ 4\mathbb{E}\| \int_0^t U(t, s) \int_0^s Q(s-\zeta) [F_2(\zeta, Z(\zeta)) - F_2(\zeta, Z^*(\zeta))] d\zeta ds \parallel^2\]
\[+ 4\mathbb{E}\| \int_0^t U(t, s) \int_0^s R(s-\zeta) [G(\zeta, Z(\zeta)) - G(\zeta, Z^*(\zeta))] dV(\zeta) ds \parallel^2\]
\[\leq 4M^2 \mathbb{E}\| Z(0) - Z^*(0) \parallel^2 + 4\Theta_2 \times \sup_{t \in \mathbb{R}} \mathbb{E}\| Z(t) - Z^*(t) \parallel^2,\]

Thus, for \( t > 0 \), we obtain
\[
\mathbb{E}\| Z(t) - Z^*(t) \parallel^2 \leq \frac{4M^2}{1 - 4\Theta_2} \mathbb{E}\| Z(0) - Z^*(0) \parallel^2.
\]

For \( \epsilon > 0 \), choosing \( 0 < \eta < \epsilon \frac{1 - 4\Theta_2}{4M^2} \), we obtain
\[
\mathbb{E}\| Z(0) - Z^*(0) \parallel^2 < \eta \implies \mathbb{E}\| Z(t) - Z^*(t) \parallel^2 \leq \epsilon, \text{ for all } t > 0.
\]

According to Definition 3.9, we can conclude that (1.1) has a unique square-mean \((\mu_1, \mu_2)\)-s.p.a.p. mild solution which is stable in square-mean sense. This completes the proof. \( \square \)
4. Examples of applications

In this section, we provide some examples to illustrate the practical usefulness of our results established in the preceding section. We can use $\mathcal{W}(t)dt = dB(t)$, where the derivative is taken in a stochastic sense. It is evident that periodicity is a very idealistic situation, and several systems may not show exactly periodic behavior. To capture such behavior, other generalized functions may be used, such as pseudo-almost periodic, pseudo-almost periodic, etc. This paper analyzes a more general class of functions that can capture the not-so-usual dynamics of several systems.

**Example 4.1.** Let us consider the generalized stochastic equation with time varying coefficient with Dirichlet boundary conditions,

$$
\frac{\partial}{\partial t} u(t, x) = -a(t) \frac{\partial^2}{\partial x^2} u(t, x) + f(t, x) + b(t)u(t, x) + \frac{h}{dt}dB(t),
$$

$$
\begin{aligned}
&u(t, 0) = u(t, \pi) = 0, u(0, x) = u_0(x), \quad x \in (0, \pi), \quad t \in \mathbb{R}^+.
\end{aligned}
$$

Let us define $\mathcal{E} = L^2(0, \pi)$ and $A(t)v = -a(t)Av$, $Av = v''$ with $D(A) = H^2(0, \pi) \cap H^1_0(0, \pi)$. It is well known that $A$ generates an analytic semigroup $T(t) : t \geq 0$. Thus $U(t, s) = e^{-\int_s^t a(\xi)d\xi}T(t-s)$. As the semigroup $T(t)$ is bounded, we obtain the condition $\|U(t, s)\| \leq Me^{-a(t-s)}$. Let $u(t, x) = u(t)x$, after these setup, the above problem can be written in the abstract form

$$
du = (A(t)u + F_1(t, u))dt + hdB(t) \quad \text{or} \quad \frac{du}{dt} = (A(t)u + F_1(t, u)) + h\mathcal{W}(t).
$$

We assume that the functions $f, b$ and $h$ are $(\mu_1, \mu_2)$ s.p.a.p. In particular, we can consider

$$
a(t) = c_1e^{-t} + |\sin t|, \quad b(t) = c_2 \frac{1}{1+t^2} + c_4 (\sin t + \sin \sqrt{2}t).
$$

We see that

$$
E\|F_1(t, x) - F_1(t, y)\| \leq E\|f(t, x) - f(t, y)\| + |b(t)|E\|x - y\|.
$$

So under the assumption that $f$ is Lipschitz, we obtain $F_1$ Lipschitz, here $L_1 = L_f + (2c_4 + c_2)$. Also $k = (1 + c_1)$. Thus we can always choose constants $c_1, c_2, c_4$, so that the required condition of Theorem holds. Hence the existence and uniqueness of $(\mu_1, \mu_2)$-s.p.a.p. solution is ensured.

**Example 4.2** (Stochastic logistic model with distributed delay). Let us consider the stochastic logistic model with distributed delay and time varying rate,

$$
dx(t) = \left( x(t)(-a(t) - b(t)x(t)) + \int_{-\infty}^t k(t-s)x(s)ds \right)dt + \sigma dB(t).
$$

We assume that $a(\cdot), b(\cdot)$ are $(\mu_1, \mu_2)$-s.p.a.p. and $k(\cdot) \in L^1([0, \infty))$. In this case, we can see $A(t) = a(t)$, $F_1(t, x(t)) = -b(t)x^2(t)$, $Q(t) = k(t)$, $F_2(t, x(t)) = x(t)$, $R(t) = \sigma$, $G = 1$. In this case, we can see that $U(t, s) = e^{\int_s^t a(\xi)d\xi}$. To satisfy the required condition, we assume that $a(t)$ is positive and bounded. Hence $\|U(t, s)\| \leq e^{-a(t-s)}$ for some positive constant $a$. We choose

$$
a(t) = c_1e^{-t} + |\sin t|, \quad b(t) = c_2 \frac{1}{1+t^2} + c_4 (\sin t + \sin \sqrt{2}t), \quad k(t) = e^{-t}.
$$
We can see in Figures 1 and 2 that the solution is \((\mu_1, \mu_2)\)-s.p.a.p. One can see that fluctuations are less when drift coefficient \(\sigma\) is small.

\[
c_1 = 2, c_2 = 1, c_4 = 1, \sigma = 3
\]

\[
c_1 = 2, c_2 = 1, c_4 = 1, \sigma = 0.2
\]

**Figure 1.** Numerical solutions of (4.2) for different parameters.

**Example 4.3** (Stochastic cellular neural networks with distributed delays). Let us consider the system of stochastic differential equations with distributed delays

\[
dx_i(t) = \left( -c_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij} \int_{-\infty}^{t} k_{ij}(t-s)g_j(x_j(s))ds + I_i \right) dt + \sigma_i dB(t). \tag{4.3}
\]

In model (4.3), \(n\) correspond to the number of neurons in the network. For \(i, j = 1, \ldots, n\); \(x_i\) represent the \(i^{th}\) neuron state, \(c_i(t)\) represent the decay rate,
\( c_1 = 0.2, \ c_2 = 0.1, \ c_4 = 1, \ \sigma = 3 \)

\( c_1 = 0.2, \ c_2 = 0.1, \ c_4 = 1, \ \sigma = 0.2 \)

**Figure 2.** Numerical solutions of (4.2) for different parameters.

\( g_j \) represent the activation function of the \( j^{th} \) neuron, \( I_i \) represent the external input in the \( i^{th} \) neuron, \( b_{ij} \) represent the connection weight and the kernel function \( k_{ij} \in L^1([0, \infty)) \). For more details on neural networks, we refer to [18, 34] and references therein. In this equation, we suppose that \( c_i(\cdot) \) are positive bounded function. Also, we choose

\[
c_i(t) = a_i e^{-t} + |\sin t|, \quad g_j(x_j(t)) = \left( a_i \frac{1}{1 + t^2} + c_4 ((\sin t + \sin \sqrt{2}t) x_j(t), \right)
\]

and \( k_{ij}(t) = e^{-t} \). The corresponding plots for \( i = 1, 2 \) are depicted in Figures 3 and 4. One can clearly see that the graphs show \((\mu_1, \mu_2)\)-s.p.a.p. behavior. In this case also, one can see that fluctuations are less when drift coefficients \( \sigma_1, \sigma_2 \) are small.
\( \alpha_1 = 2, \alpha_2 = 1, \sigma_1 = 3, \sigma_2 = 2. \)

\( \alpha_1 = 2, \alpha_2 = 1, \sigma_1 = 0.3, \sigma_2 = 0.2 \)

**Figure 3.** Numerical solutions of (4.3) for different parameters.

**Acknowledgements.** We are thankful to the reviewers for their constructive comments and suggestions which helped us to improve this article.

**References**


\(a_1 = 0.2, a_2 = 0.1, c_4 = 1, \sigma_1 = 3, \sigma_2 = 2\)

\(a_1 = 0.2, a_2 = 0.1, c_4 = 1, \sigma_1 = 0.3, \sigma_2 = 0.2\)

**Figure 4.** Numerical solutions of (4.3) for different parameters.


[22] T. Diagana, K. Ezzinbi, M. Miraoui; Pseudo-almost periodic and pseudo-almost automorphic solutions to some evolution equations involving theoretical measure theory, Cubo (Temuco), 16(2) (2014), 01-32.


Moez Ayachi
Laboratory of Mathematics and Applications (LR17ES11), Faculty of Sciences of Gabes, University of Gabes, Gabes 6072, Tunisia
Email address: Moez.Ayachi@fsg.rnu.tn

Syed Abbas (corresponding author)
School of Mathematical and Statistical Sciences, Indian Institute of Technology Mandi, Mandi, H.P., 175005, India
Email address: abbas@iitmandi.ac.in