NODAL SOLUTIONS FOR NONLINEAR SCHRÖDINGER SYSTEMS

XUE ZHOU, XIANGQING LIU

Abstract. In this article we consider the nonlinear Schrödinger system

\[-\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j, \quad \text{in } \Omega,\]

\[u_j(x) = 0, \quad \text{on } \partial \Omega, \quad j = 1, \ldots, k,\]

where \(\Omega \subseteq \mathbb{R}^N (N = 2, 3)\) is a bounded smooth domain, \(\lambda_j > 0, j = 1, \ldots, k,\)
\(\beta_{jj}\) are constants satisfying \(\beta_{jj} > 0, \beta_{ij} = \beta_{ji}, 1 \leq i < j \leq k.\) The
existence of sign-changing solutions is proved by the truncation method and
the invariant sets of descending flow method.

1. Introduction

We consider the nonlinear Schrödinger system

\[-\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j, \quad \text{in } \Omega,\]

\[u_j(x) = 0, \quad \text{on } \partial \Omega, \quad j = 1, \ldots, k,\]

where \(\Omega \subseteq \mathbb{R}^N (N = 2, 3)\) is a bounded domain with smooth boundary, and \(\lambda_j > 0,\)
\(\beta_{jj} > 0, 1 \leq j \leq k, \beta_{ij} = \beta_{ji}, 1 \leq i < j \leq k\) are constants.

This type of coupled systems, also known as Gross-Pitaevskii equations, have
applications in many physical problems such as nonlinear optics and multispecies
Bose-Einstein condensates \[5 \ 13\]. Physically, \(\beta_{jj}, \beta_{ij} (i \neq j)\) are the intraspecies
and interspecies scattering lengths respectively. In the physics literature, the signs
of the coupling constants \(\beta_{ij}\) being positive or negative determine the nature of
the system being attractive or repulsive. In the repulsive case \((\beta_{ij} < 0, i \neq j,\)
i, j = 1, \ldots, k\), the components tend to segregate with each other leading to phase
separations. These phenomena have been documented in experiments as well as in
numeric simulations; see \[4 \ 17\] and references therein. Mathematical work has been
done extensively in recent years, refer the reader to \[1 \ 3 \ 7 \ 9 \ 14 \ 15 \ 16 \ 19\] for the
existence theory and the studies of qualitative property of solutions to attractive
and repulsive systems.
Over the years there have been systematic studies on nodal solutions for scalar equations by using a combination of minimax methods and the method of invariant sets of gradient flows. We refer the reader to \cite{2, 6, 13}. However, most of the methods in treating scalar equations are not applicable directly to systems. In \cite{14, 15} a construction of invariant sets has been developed to locate multiple nontrivial solutions, but without giving any information about nodal property of the components of solutions. Compared with scalar equations, there are many new challenges for coupled equations in dealing with the existence of multiple solutions, in particular multiple sign-changing solutions. An attempt was made in \cite{10, 11} for establishing an abstract framework to deal with sign-changing solutions for systems that share some of the above features. The authors in \cite{10, 11} developed the method of multiple invariant sets of decreasing flow. In \cite{10} for the subcritical case infinitely many sign-changing solutions were established. Specially, Chen, Lin and Zou \cite{5} proved the existence of multiple sign-changing (i.e., both two components change sign) and semi-nodal solutions (i.e., one component changes sign and the other one is positive) for coupled Schrödinger equations for the case of $k = 2$, $\beta_{12} = \beta_{21} = \beta > 0$. Motivated by the works we mentioned above, in this paper we consider the existence of sign-changing solutions for the system \eqref{system} in the general case, by using the method of invariant sets of decreasing flow (see \cite{10}) and the truncation method (see \cite{12}).

We assume that

1. $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, $k \geq 2$, $\lambda_j > 0$ for $j = 1, \ldots, k$.
2. $\beta_{jj} > 0$, $\beta_{ij} = \beta_{ji} \leq 0$ for $1 \leq i < j \leq k$.

Solutions of \eqref{system} correspond to critical points of the functional

$$I(U) = \frac{1}{2} \int_{\Omega} \sum_{j=1}^{k} (|\nabla u_j|^2 + \lambda_j u_j^2) \, dx - \frac{1}{4} \int_{\Omega} \sum_{i,j=1}^{k} \beta_{ij} u_i^2 u_j^2 \, dx$$

for $U = (u_1, \ldots, u_k) \in X = H_0^1(\Omega) \times \cdots \times H_0^1(\Omega)$, the $k$-fold product of $(H_0^1(\Omega))^k$.

We shall use the equivalent inner products

$$(u, v)_j = \int_{\Omega} (\nabla u \nabla v + \lambda_j uv) \, dx, \quad j = 1, \ldots, k$$

and the induced norm $\| \cdot \|_j$. The inner product

$$(U, V) = \sum_{j=1}^{k} (u, v)_j, \quad U = (u_1, \ldots, u_k), \, V = (v_1, \ldots, v_k),$$

gives rise to a norm $\| \cdot \|$ on $X$.

Firstly, we introduce the following perturbation problem. We assume $U = (u_1, \ldots, u_k), \, \varepsilon \in \mathbb{R}$ is a small parameter, $F(U, \varepsilon)$, $\frac{\partial F}{\partial u_j}(U, \varepsilon)$ are continuous functions, and $F(U, \varepsilon) = F(-U, \varepsilon)$. For $\varepsilon = 0$, we understand

$$F(U, 0) = 0, \quad \frac{\partial F}{\partial u_j}(U, 0) = 0.$$

Then we consider the perturbed problem

$$-\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j + \frac{\partial F}{\partial u_j}(U, \varepsilon), \quad \text{in } \Omega,$$

$$u_j(x) = 0, \quad \text{on } \partial \Omega, \, j = 1, \ldots, k.$$

(1.2)
Here are our main results.

**Theorem 1.1.** Assume (A1), (A2) hold. Then system (1.1) has infinitely many solutions with each component being sign-changing.

**Theorem 1.2.** Assume (A1), (A2) hold and let \( l \in \mathbb{N}^+ \). Then there exists \( \varepsilon_l > 0 \) such that for \( |\varepsilon| \leq \varepsilon_l \), the system (1.2) has \( l \) pairs of sign-changing solutions.

**Corollary 1.3.** For each \( l \in \mathbb{N}^+ \), there exists \( \beta_l > 0 \) such that for \( \beta_{ij} = \beta_{ji} \leq \beta_l \) with \( 1 \leq i < j \leq k \), system (1.1) has at least \( l \) pairs of sign-changing solutions.

Note that we do not assume any growth conditions for the perturbation function \( F \). To apply critical point theorem [10, 11], we firstly have the following truncated function; the idea comes from [12]. For \( M > 0 \), we define

\[
F_M(U, \varepsilon) = F\left(f_M(|U|) \frac{U}{|U|}\right),
\]

where \( f_M \) is a monotonic smooth function, satisfying \( f_M(t) = t \) if \( t \leq M \), \( f_M(t) = M + \frac{1}{2} \) if \( t \geq M \). Then we consider the truncated system

\[
-\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j + \frac{\partial F_M(U, \varepsilon)}{\partial u_j}, \quad \text{in } \Omega,
\]

\[
u_j(x) = 0, \quad \text{on } \partial \Omega, \quad j = 1, \ldots, k.
\]

If \( U = (u_1, \ldots, u_k) \) is a solution of (1.3), and there exists \( M > 0 \) such that \( |U(x)| < M \) for all \( x \in \overline{\Omega} \), then \( U \) is also a solution of the perturbed problem (1.2). System (1.3) has a variational structure given by the functional

\[
I_M(U) = I(U) - \int_{\Omega} F_M(U, \varepsilon) \, dx
= \frac{1}{2} \int_{\Omega} \sum_{j=1}^{k} (|\nabla u_j|^2 + \lambda_j u_j^2) \, dx - \frac{1}{4} \int_{\Omega} \sum_{i,j=1}^{k} \beta_{ij} u_i^2 u_j^2 \, dx - \int_{\Omega} F_M(U, \varepsilon) \, dx.
\]

This article organized as follows. In Section 2, we study the truncated functional \( I_M \), and construct a sequence of critical values for \( I_M \) by using the method of multiple invariant sets of descending flow. In Section 3, we obtain the sign-changing solutions of the perturbed problem (1.2), then we obtain the main result.

Throughout this article, we use \( \| \cdot \|_{L^p} \) and \( \| \cdot \| \) to denote the norms of \( L^p \) and \( X \), respectively. \( c, c_1, \ldots \) denote constants that are independent of the sequences in the arguments but maybe different from line to line, and \( c(\cdot) \) will be used to indicate the dependency of the constant \( c \) on the relevant quantity.

2. **Critical points of the truncated functional \( I_M \)**

To obtain sign-changing critical points of \( I_M \), we apply an abstract critical point theorem (Theorem 2.1) to the truncated functional \( I_M \).

**Theorem 2.1.** Let \( X \) be a Banach space, \( f \) be an even \( C^1 \)-functional on \( X \), \( A \) be an odd, continuous mapping from \( X \) to \( X \), and \( P_j, Q_j, j = 1, \ldots, k \) be open convex subsets of \( X \) with \( Q_j = -P_j \). Denote \( W = \bigcup_{j=1}^{k} (P_j \cup Q_j) \), \( \Sigma = \cap_{j=1}^{k} (\partial P_j \cap \partial Q_j) \). Assume

(A3) \( f \) satisfies the Palais-Smale condition.
(A4) \( c^* = \inf_{x \in \Sigma} f(x) > 0 \).
(A5) For each \( b_0 > 0 \) and \( c_0 > 0 \), there exists \( b = b(b_0, c_0) \), such that if \( |f(x)| \leq c_0 \), \( \|Df(x)\| \geq b_0 \), then
\[
\langle Df(x), x - Ax \rangle \geq b\|x - Ax\| > 0.
\]
(A6) \( A(\partial P_j) \subset P_j \), \( A(\partial Q_j) \subset Q_j \), \( j = 1, \ldots, k \).

We define
\[
\Gamma_j = \{ E \subset X : E \text{ is compact}, -E = E, \gamma(E \cap \sigma^{-1}(\Sigma)) \geq j \text{ for } \sigma \in \Lambda \}.
\]
\[
\Lambda = \{ \sigma \in C(X, X) : \sigma \text{ is odd}, \sigma(P_j) \subset P_j, \sigma(Q_j) \subset Q_j, j = 1, \ldots, k, \sigma(x) = x \text{ if } f(x) < 0 \}
\]
where \( \gamma = \gamma(E) \) denotes the genus of a symmetric set \( E \)
\[
\gamma = \min \{ n : \text{there is an odd map } \varphi^{(j)} : E \to \mathbb{R}^n \setminus \{0\} \}.
\]
We assume that
(A7) \( \Gamma_j \) is nonempty for \( j = 1, 2, \ldots, k \).

Then we define
\[
c_j = \inf_{E \in \Gamma_j} \sup_{x \in E \setminus \emptyset} f(x), \quad j = 1, 2, \ldots,
\]
\[
K_c = \{ x \in X : Df(x) = 0, f(x) = c \}, \quad K^*_c = K_c \setminus W.
\]
Then
1. \( c_j \geq c_s, K^*_{c_j} \neq \emptyset \) for \( j = 1, 2, \ldots, k \).
2. \( c_j \to +\infty \), as \( j \to \infty \).
3. If \( c_j = c_{j+1} = \cdots = c_{j+l-1} = c \), then \( \gamma(K^*_c) \geq l \).

**Lemma 2.2.** \( I_M \) is a \( C^1 \)-functional on \( X \), and satisfies the Palais-Smale condition.

**Proof.** It is easy to verify that \( I_M \) is a \( C^1 \)-functional. Also, for \( \Phi = (\varphi_1, \ldots, \varphi_k) \in X \), we have
\[
\langle DI_M(U), \Phi \rangle = \int_{\Omega} \sum_{j=1}^{k} (\nabla u_j \nabla \varphi_j + \lambda_j u_j \varphi_j) \, dx - \int_{\Omega} \sum_{i,j=1}^{k} \beta_{ij} u_i^2 u_j \varphi_j \, dx - \int_{\Omega} \sum_{j=1}^{k} \frac{\partial F_M}{\partial u_j}(U, \varepsilon) \varphi_j \, dx,
\]
there exists an arbitrary small constant \( \varepsilon_M \), such that for \( |\varepsilon| \leq \varepsilon_M \), we have
\[
I_M(U) - \frac{1}{4} \langle DI_M(U), U \rangle
\]
\[
= \frac{1}{4} \int_{\Omega} \sum_{j=1}^{k} (|\nabla u_j|^2 + \lambda_j u_j^2) \, dx - \int_{\Omega} (F_M(U, \varepsilon) - \frac{1}{4} \sum_{j=1}^{k} \frac{\partial F_M}{\partial u_j}(U, \varepsilon) u_j) \, dx
\]
\[
\geq \frac{1}{4} \|U\|^2 - c.
\]
Then any Palais-Smale sequence of \( I_M \) is bounded in \( X \). Let \( U_n = (u_{n,1}, \ldots, u_{n,k}) \in X \) be a Palais-Smale sequence of the functional \( I_M \). Notice that the imbedding
$H^1_0(\Omega) \hookrightarrow L^4(\Omega)$ is compact and we can assume that $U_n \to U$ in $L^4(\Omega)$. Then we have

$$
\int_{\Omega} \sum_{j=1}^k ([\nabla(u_{n,j} - u_{m,j})]^2 + \lambda_j (u_{n,j} - u_{m,j})^2) \, dx
$$

$$
= \langle DI_M(U_n) - DI_M(U_m), U_n - U_m \rangle + \int_{\Omega} \sum_{j=1}^k \beta_{ij} u_{n,j}^2 (u_{n,j} - u_{m,j}) \, dx
$$

$$
- \int_{\Omega} \sum_{j=1}^k \beta_{ij} u_{m,j}^2 (u_{n,j} - u_{m,j}) \, dx
$$

$$
+ \int_{\Omega} \sum_{j=1}^k \left( \frac{\partial F_M(U_n, \varepsilon)}{\partial u_j} - \frac{\partial F_M(U_m, \varepsilon)}{\partial u_j} \right) (u_{n,j} - u_{m,j}) \, dx
$$

$$
\leq o(1) + c\|U_n\|_{L^4(\Omega)}^3 \left( \int_{\Omega} \sum_{j=1}^k (u_{n,j} - u_{m,j})^4 \, dx \right)^{1/4}
$$

$$
+ c\|U_m\|_{L^4(\Omega)}^3 \left( \int_{\Omega} \sum_{j=1}^k (u_{n,j} - u_{m,j})^4 \, dx \right)^{1/4}
$$

$$
+ \int_{\Omega} \sum_{j=1}^k \frac{\partial F_M(U_n, \varepsilon)}{\partial u_j} - \frac{\partial F_M(U_m, \varepsilon)}{\partial u_j} |u_{n,j} - u_{m,j}| \, dx
$$

$$
\leq o(1) + c\|U_n - U_m\|_{L^4(\Omega)} \to 0, \quad \text{as} \quad n, m \to \infty.
$$

Therefore, we conclude that up to a subsequence a Palais-Smale sequence $U_n$ is a Cauchy sequence in $X$, hence a convergent sequence. □

**Definition 2.3.** An odd and continuous operator $A : U = (u_1, \ldots, u_k) \in X \mapsto V = (v_1, \ldots, v_k) = AU \in X$ is defined by the system

$$
\int_{\Omega} (\nabla v_j \nabla \varphi_j + \lambda_j v_j \varphi_j) \, dx - \int_{\Omega} \sum_{i=1, i \neq j}^k \beta_{ij} u_i^2 v_j \varphi_j \, dx
$$

$$
= \int_{\Omega} \beta_{jj} u_j^3 \varphi_j \, dx + \int_{\Omega} \frac{\partial F_M}{\partial u_j}(U, \varepsilon) \varphi_j \, dx,
$$

For $j = 1, \ldots, k$ and $\Phi = (\varphi_1, \ldots, \varphi_k) \in X$.

**Lemma 2.4.** The operator $A$ is well-defined and continuous.

**Proof.** Note that $V = AU$ can be obtained by solving the minimization problem

$$
\inf \{ G(V) : V \in X \}
$$

where

$$
G(V) = \frac{1}{2} \int_{\Omega} \sum_{j=1}^k (|\nabla v_j|^2 + \lambda_j v_j^2) \, dx - \frac{1}{2} \int_{\Omega} \sum_{i=1, i \neq j}^k \beta_{ij} u_i^2 v_j^2 \, dx
$$

$$
- \int_{\Omega} \sum_{j=1}^k \beta_{jj} u_j^3 v_j \, dx - \int_{\Omega} \sum_{j=1}^k \frac{\partial F_M}{\partial u_j}(U, \varepsilon) v_j \, dx.
$$
Let \( V = AU, \bar{V} = A\bar{U}, \bar{V} = (\bar{v}_1, \ldots, \bar{v}_k), \bar{U} = (\bar{u}_1, \ldots, \bar{u}_k) \). By (2.3), we have
\[
\|V - \bar{V}\|^2 = \int_\Omega \sum_{j=1}^k (|\nabla (v_j - \bar{v}_j)|^2 + \lambda_j (v_j - \bar{v}_j)^2) \, dx
\]
\[
= \int_\Omega \sum_{i,j=1, i \neq j}^k \beta_{ij} (u_i^2 v_j - \bar{u}_i^2 \bar{v}_j)(v_j - \bar{v}_j) \, dx + \int_\Omega \sum_{j=1}^k \beta_{jj} (u_j^3 - \bar{u}_j^3)(v_j - \bar{v}_j) \, dx
\]
\[
+ \int_\Omega \sum_{j=1}^k \left( \frac{\partial F_M}{\partial u_j}(U, \varepsilon) - \frac{\partial F_M}{\partial u_j}(\bar{U}, \varepsilon) \right) (v_j - \bar{v}_j) \, dx
\]
\[
\leq c \int_\Omega \sum_{i,j=1, i \neq j}^k |u_i^2 - \bar{u}_i^2| |v_j| |v_j - \bar{v}_j| \, dx + c \int_\Omega \sum_{j=1}^k |u_j^3 - \bar{u}_j^3| |v_j - \bar{v}_j| \, dx
\]
\[
+ \int_\Omega \sum_{j=1}^k \left| \frac{\partial F_M}{\partial u_j}(U, \varepsilon) - \frac{\partial F_M}{\partial u_j}(\bar{U}, \varepsilon) \right| |v_j - \bar{v}_j| \, dx
\]
\[
\leq c(\|U - \bar{U}\| \|V - \bar{V}\| + \| \frac{\partial F_M}{\partial u_j}(U, \varepsilon) - \frac{\partial F_M}{\partial u_j}(U, \varepsilon) \| \|V - \bar{V}\|).
\]

hence \( AU - A\bar{U} = V - \bar{V} \rightarrow 0 \) as \( U \rightarrow \bar{U} \) in \( X \). \( \square \)

**Lemma 2.5.** For each \( b_0, c_0 > 0 \), then the following property holds: if \( |I_M(U)| \leq c_0 \) and \( \|DI_M(U)\| \geq b_0 \), then there exists \( b = b(b_0, c_0) \) such that
\[
\langle DI_M(U), U - AU \rangle \geq b \|U - AU\| > 0.
\]

**Proof.** We have
\[
\langle DI_M(U), \Phi \rangle = \int_\Omega \sum_{j=1}^k (\nabla (u_j - v_j)) \nabla \varphi_j + \lambda_j (u_j - v_j) \varphi_j \, dx - \int_\Omega \sum_{i,j=1, i \neq j}^k \beta_{ij} u_i^2 (u_j - v_j) \varphi_j \, dx - \int_\Omega \sum_{i,j=1, i \neq j}^k \beta_{ij} u_i^2 (u_j - v_j) \varphi_j \, dx
\]
\[
= \langle U - V, \Phi \rangle - \int_\Omega \sum_{i,j=1, i \neq j}^k \beta_{ij} u_i^2 (u_j - v_j) \varphi_j \, dx
\]
(2.4)

for \( \Phi = (\varphi_1, \ldots, \varphi_k) \in X \). By using \( \Phi = U - V \) in (2.4), we obtain
\[
\langle DI_M(U), U - V \rangle = \|U - V\|^2 - \int_\Omega \sum_{i,j=1, i \neq j}^k \beta_{ij} u_i^2 (u_j - v_j)^2 \, dx.
\]

Notice that if \( \beta_{ij} = \beta_{ji} \leq 0 \) for \( 1 \leq i < j \leq k \), then
\[
\langle DI_M(U), U - V \rangle \geq \|U - V\|^2
\]
(2.5)

and
\[
\langle DI_M(U), U - V \rangle \geq -\int_\Omega \sum_{i,j=1, i \neq j}^k \beta_{ij} u_i^2 (u_j - v_j)^2 \, dx.
\]
(2.6)
It follows from (2.4) and (2.6) that

\[
|\langle DI_M(U), \Phi \rangle| = |\langle U - V, \Phi \rangle - \int_{\Omega} \sum_{i,j=1, i \neq j}^{k} \beta_{ij} u_i^2(u_j - v_j) \phi_j \, dx |
\]

\[
\leq \|U - V\| \|\Phi\| + \left( - \int_{\Omega} \sum_{i,j=1, i \neq j}^{k} \beta_{ij} u_i^2(u_j - v_j)^2 \, dx \right)^{1/2}
\]

\[
\times \left( - \int_{\Omega} \sum_{i,j=1, i \neq j}^{k} \beta_{ij} u_i^2 \varphi_j^2 \, dx \right)^{1/2}
\]

\[
\leq \|U - V\| \|\Phi\| + c\|U\|_{L^4(\Omega)} \|\Phi\|_{L^4(\Omega)} \langle DI_M(U), U - V \rangle^{1/2}
\]

which implies that

\[
\|DI_M(U)\| \leq \|U - V\| + c\|U\| \langle DI_M(U), U - V \rangle^{1/2}.
\]  \hspace{1cm} (2.7)

There exists a small constant \( \varepsilon_M \), so that for \( |\varepsilon| \leq \varepsilon_M \), by (1.4) and (2.4), we have

\[
I_M(U) - \frac{1}{4} \langle U - V, U \rangle
\]

\[
= I_M(U) - \frac{1}{4} \langle DI_M(U), U \rangle - \frac{1}{4} \int_{\Omega} \sum_{i,j=1, i \neq j}^{k} \beta_{ij} u_i^2 u_j(u_j - v_j) \, dx
\]

\[
= \frac{1}{4} \|U\|^2 + \int_{\Omega} \left( \frac{1}{4} \sum_{j=1}^{k} \frac{\partial F_M}{\partial u_j}(U, \varepsilon) u_j - F_M(U, \varepsilon) \right) \, dx
\]

\[
- \frac{1}{4} \int_{\Omega} \sum_{i,j=1, i \neq j}^{k} \beta_{ij} u_i^2 u_j(u_j - v_j) \, dx
\]

\[
\geq \frac{1}{4} \|U\|^2 - \frac{1}{4} \int_{\Omega} \sum_{i,j=1, i \neq j}^{k} \beta_{ij} u_i^2 u_j(u_j - v_j) \, dx - c.
\]  \hspace{1cm} (2.8)

So by (2.8), we obtain

\[
\|U\|^2 \leq c(1 + |I_M(U)|) + c\|U - V, U\| + c \int_{\Omega} \sum_{i,j=1, i \neq j}^{k} \beta_{ij} u_i^2 u_j(u_j - v_j) \, dx
\]  \hspace{1cm} (2.9)

\[
\leq c(1 + |I_M(U)|) + c\|U - V\|^2 + \frac{1}{4} \|U\|^2 + c\|U\|_{L^4(\Omega)}^2 \langle DI_M(U), U - V \rangle^{1/2}.
\]

Given a positive constant \( a \), if

\[
\langle DI_M(U), U - V \rangle \geq a^2,
\]

then by (2.6) we can easily obtain

\[
\langle DI_M(U), U - V \rangle \geq a\|U - V\| > 0.
\]

The conclusion holds; if not, let

\[
\langle DI_M(U), U - V \rangle \leq a^2,
\]  \hspace{1cm} (2.10)

by (2.9) and (2.10), we have

\[
\|U\|^2 \leq c(1 + |I_M(U)|) + \|U - V\|^2 + c_0 a\|U\|^2.
\]  \hspace{1cm} (2.11)
Hence, taking $a$ such that $c_o a \leq 1/2$, then we have
\[ \|U\|^2 \leq c(1 + |I_M(U)| + \|U - V\|^2). \] (2.12)

Substituting (2.12) into (2.7), we obtain
\[ \|DI_M(U)\| \leq \|U - V\| + c(1 + |I_M(U)| + \|U - V\|^2) \frac{1}{2}(DI_M(U), U - V)^{1/2} \] (2.13)
\[ \leq \|U - V\| + \frac{1}{2}\|DI_M(U)\| + c(1 + |I_M(U)| + \|U - V\|^2)\|U - V\|. \]

Therefore
\[ \|DI_M(U)\| \leq c(1 + |I_M(U)| + \|U - V\|^2)\|U - V\|. \]

If $|I_M(U)| \leq c_0$ and $\|DI_M(U)\| \geq b_0 > 0$, we deduce that there exists $b = b(b_0, c_0)$ such that $\|U - V\| > b$. So it follows from (2.5) that
\[ \langle DI_M(U), U - AU \rangle \geq b\|U - AU\| > 0. \]

Let $P_j, Q_j$ for $j = 1, \ldots, k$ be open convex subsets of $X$, defined by
\[ P_j = P_j(\delta) = \{U = (u_1, \ldots, u_k) \in X : \|u_j\|_{L^1(\Omega)} < \delta\}, \]
\[ Q_j = Q_j(\delta) = \{U = (u_1, \ldots, u_k) \in X : \|u_j\|_{L^1(\Omega)} < \delta\}. \]

**Lemma 2.6.** There exist $\delta > 0$ and $\varepsilon_M > 0$ such that for $|\varepsilon| \leq \varepsilon_M$, it holds that
\[ A(\partial P_j) \subset P_j, \ A(\partial Q_j) \subset Q_j, \text{ for } j = 1, \ldots, k. \]

**Proof.** Choose $\Phi = V^+ = (v_1^+, \ldots, v_k^+)$ as test function in (2.3), we have
\[ \int_{\Omega} (|\nabla v_j^+|^2 + \lambda_j(v_j^+)^2) \, dx - \int_{\Omega} \sum_{i=1,i\neq j}^k \beta_{ij} u_i^2(v_j^+)^2 \, dx \]
\[ = \int_{\Omega} \beta_{jj} u_j^2 v_j^+ \, dx + \int_{\Omega} \frac{\partial F_M}{\partial u_j}(U, \varepsilon) v_j^+ \, dx \]
\[ \leq c \left( \int_{\Omega} (u_j^+)^3 v_j^+ \, dx + \int_{\Omega} \left| \frac{\partial F_M}{\partial u_j}(U, \varepsilon) \right| v_j^+ \, dx \right). \]

Then
\[ \|v_j^+\|^2_{L^2(\Omega)} \leq c_1 \|u_j^+\|^3_{L^1(\Omega)} \|v_j^+\|_{L^2(\Omega)} + c_2 \left| \frac{\partial F_M}{\partial u_j}(U, \varepsilon) \right|_{L^\infty(\Omega)} \|v_j^+\|_{L^1(\Omega)}. \] (2.14)

Take $\delta > 0$ such that $c_1\delta^2 \leq 1/4$ and choose $\varepsilon_M > 0$, such that for $|\varepsilon| \leq \varepsilon_M$, $c_2 \left| \frac{\partial F_M}{\partial u_j}(U, \varepsilon) \right|_{L^\infty(\Omega)} \leq \delta/4$. Then for $U \in \partial Q_j$, $\|u_j^+\|_{L^2(\Omega)} = \delta$, we have
\[ \|v_j^+\|^2_{L^2(\Omega)} \leq \frac{1}{4} \delta \|v_j^+\|_{L^2(\Omega)} + \frac{1}{4} \delta \|v_j^+\|_{L^2(\Omega)}, \]

hence
\[ \|v_j^+\|_{L^2(\Omega)} \leq \frac{1}{2} \delta. \]

That is for $U \in \partial Q_j$, we have $V = AU \in Q_j$ and $A(\partial Q_j) \subset Q_j$, $j = 1, \ldots, k$. Similarly, $A(\partial P_j) \subset P_j$, $j = 1, \ldots, k$. \[ \square \]

**Lemma 2.7.** There exist $\delta > 0$ and $c^* > 0$, such that if $U \in \Sigma$ and $|\varepsilon| \leq \varepsilon_M$, then $I_M(U) \geq c^*$. 
Lemma 2.8. The set $\mathcal{I}(\mathcal{U}) = \{1, 2, \ldots, n\}$ is nonempty.

\textbf{Proof.} Note that
\[
I_M(U) = \frac{1}{2} \int_{\Omega} \sum_{j=1}^{k} (|\nabla u_j|^2 + \lambda_j u_j^2) \, dx - \frac{1}{4} \int_{\Omega} \sum_{j=1}^{k} \beta_{ij} u_j^2 u_i^2 \, dx - \int_{\Omega} F_M(U, \varepsilon) \, dx
\]
\[
\geq \frac{1}{2} \|U\|^2 - \frac{1}{4} \int_{\Omega} \sum_{j=1}^{k} \beta_{jj} u_j^4 \, dx - \int_{D} F_M(U, \varepsilon) \, dx
\]
\[
\geq c_1 \|U\|^2 - c_2 \|U\|^4 - \|F_M(U, \varepsilon)\|_{L^\infty(\Omega)}.
\]
For $U \in \mathcal{I} = \cap_{j=1}^{k} (\partial P_j \cap \partial Q_j)$, we have
\[
\|U\|^2_{L^4(\Omega)} = \int_{\Omega} \sum_{j=1}^{k} (|u_j^+|^4 + |u_j^-|^4) \, dx = 2k \|u_j^+\|^4_{L^4(\Omega)} = 2k \delta^4.
\]
By Lemma 2.6 taking $\delta > 0$ such that $c_2 \delta^2 \leq \frac{1}{4} c_1$, and choosing $\varepsilon_M$ such that for $|\varepsilon| \leq \varepsilon_M$, we have $\|F_M(U, \varepsilon)\|_{L^\infty(\Omega)} \leq \frac{1}{4} c_1 \delta^2$. Therefore,
\[
I_M(U) \geq c_1 \delta^2 - c_2 \delta^4 - \frac{1}{4} c_1 \delta^2 \geq \frac{1}{2} c_1 \delta^2 := \varepsilon^* > 0.
\]

Let
\[
\Gamma_j = \{E \subset X : E \text{ is compact}, -E = E, \gamma(E \cap \sigma^{-1}(\Gamma)) \geq j \text{ for } \sigma \in \mathcal{A}\},
\]
\[
\mathcal{A} = \{\sigma \in C(X, X) : \text{ odd, } \sigma(P_j) \subset P_j, \sigma(Q_j) \subset Q_j, j = 1, \ldots, k, \sigma(U) = U \text{ if } I_M(U) < 0\},
\]
and $\gamma = \gamma(E)$ is the genus of $E$,
\[
\gamma = \min\{n : \text{ there is an odd map } \varphi^{(j)} : E \to \mathbb{R}^n \setminus \{0\}\}.
\]
Now we define a sequence of critical values of the truncated functional $I_M$,
\[
c_j(M, \varepsilon) = \inf_{E \in \Gamma_j} \sup_{U \in E \setminus W} I_M(U), \quad j = 1, 2, \ldots
\]
where $W = \cup_{j=1}^{k} (P_j \cup Q_j)$.

\textbf{Lemma 2.8.} The set $\Gamma_j$ is nonempty, and there exist $d_j > 0$ independent of $M, \varepsilon$ and $\varepsilon_M > 0$, such that if $|\varepsilon| \leq \varepsilon_M$, then $c_j(M, \varepsilon) \leq d_j$.

\textbf{Proof.} Let $B^{nk}$ be the unit closed ball of $\mathbb{R}^{nk}$. Assume $n = j + k$. Denote $t \in \mathbb{R}^{nk}$ by $t = (t_1, \ldots, t_k)$ and $t_m = (t_{1m}, t_{2m}, \ldots, t_{nm}) \in \mathbb{R}^n$ for $m = 1, \ldots, k$. Let $v_{im} \in C_0^\infty(\Omega)$, $i = 1, \ldots, n$, $m = 1, \ldots, k$ be $nk$ functions in $X$ with disjoint supports. Define $\varphi^{(j)} : B^{nk} \to X$ by
\[
\varphi^{(j)}(t) = R\left(\sum_{i=1}^{n} t_{i1} v_{i1}, \ldots, \sum_{i=1}^{n} t_{ik} v_{ik}\right) \in X
\]
where $R$ is large enough such that $I(\varphi^{(j)}(t)) < -10$ for $t \in \partial B^{nk}$. Then there exists $\varepsilon_M > 0$, so that if $|\varepsilon| \leq \varepsilon_M$, then we have
\[
I_M(\varphi^{(j)}(t)) \leq I(\varphi^{(j)}(t)) + 1 < 0
\]
for $t \in \partial B^{nk}$. By [1], Lemma 5.6], we have $E_j := \varphi^{(j)}(B^{nk}) \in \Gamma_j$. Then $\Gamma_j$ is nonempty.
Next we estimate \( c_j(M, \varepsilon) \) for \(|\varepsilon| \leq \varepsilon_M\). We have

\[
c_j(M, \varepsilon) = \inf_{E \in \Gamma_j} \sup_{U \in E \setminus W} I_M(U) \leq \sup_{U \in E_j} I_M(U) \leq \sup_{U \in E_j} (I(U) + 1) := d_j. \tag*{□}
\]

3. Proof of main results

In this section, we complete the proof of Theorem 1.1 and Theorem 1.2. For fixed \( M > 0 \) and \( \varepsilon = 0 \), we will obtain the critical point \( U \) of \( I \).

**Lemma 3.1.** Assume \( DI_M(U) = 0, I_M(U) \leq L \). Then there exist \( \varepsilon_M > 0 \) and \( K = K(L) \) independent of \( M, \varepsilon \), such that for \(|\varepsilon| \leq \varepsilon_M\),

\[
\|U(x)\|_{L^\infty(\Omega)} \leq K.
\]

**Proof.** Denote \( U = (u_1, \ldots, u_k) \). By (2.2), for \(|\varepsilon| \leq \varepsilon_M\), we have

\[
L \geq I_M(U) - \frac{1}{4} \langle DI_M(U), U \rangle = \frac{1}{4} \int_\Omega \sum_{j=1}^k (|\nabla u_j|^2 + \lambda_j u_j^2) \, dx - \int_\Omega \left( F_M(U, \varepsilon) - \frac{1}{4} \sum_{j=1}^k \frac{\partial F_M}{\partial u_j}(U, \varepsilon) u_j \right) \, dx \geq \frac{1}{4} \|U\|^2 - c. \tag{3.1}
\]

We know that there exists \( C(L) > 0 \), such that \( \|U\| \leq C(L) \). Choose \( \phi = u_{jT} |u_{jT}|^{2r-2} \) as the test function in \( \langle DI_M(u_j), \phi \rangle = 0 \), where \( r \geq 1, T > 1, \) and \( u_{jT}(x) = \pm T \) if \( \pm u_j(x) \geq T, u_{jT}(x) = u_j(x) \) if \( |u_j(x)| \leq T \). We have

\[
\int_\Omega (\nabla u_j \nabla \phi + \lambda_j u_j \phi) \, dx = \int_\Omega \sum_{i=1}^k \beta_{ij} u_i^2 u_j \phi \, dx + \int_\Omega \frac{\partial F_M}{\partial u_j}(U, \varepsilon) \phi \, dx. \tag{3.2}
\]

By (3.2), it is easy to obtain the inequality

\[
\int_\Omega \nabla u_j \nabla \phi \, dx \leq \int_\Omega \beta_{jj} u_j^3 \phi \, dx + \int_\Omega \left| \frac{\partial F_M}{\partial u_j}(U, \varepsilon) \phi \right| \, dx. \tag{3.3}
\]

Firstly, we estimate the left-hand side of (3.3),

\[
\int_\Omega \nabla u_j \nabla \phi \, dx \geq (2r - 1) \int_\Omega |\nabla u_{jT}|^2 |u_{jT}|^{2r-2} \, dx \geq \frac{2r - 1}{r^2} \int_\Omega |\nabla |u_{jT}|^r|^2 \, dx \geq \frac{(2r - 1)}{r^2} \left( \int_\Omega (|u_{jT}|^r)^{2^*} \, dx \right)^{2/2^*}. \tag{3.4}
\]
Let $M > 0$, there exists $\varepsilon_M$ such that for $|\varepsilon| \leq \varepsilon_M$, we have $\|\partial F_m / \partial u_j (U, \varepsilon)\|_{L^\infty(\Omega)} < 1$. Then the right-hand side of (3.3) satisfies

$$\int_\Omega \beta_j u_j^3 \phi \, dx + \int_\Omega \left| \frac{\partial F_m}{\partial u_j} (U, \varepsilon) \right| \phi \, dx$$

$$\leq c \left( \int_\Omega |u_j|^3 |u_j|^2 \, dx + \int_\Omega |u_j|^2 \, dx \right)$$

$$\leq c \left( \int_\Omega (1 + |u_j|^3) |u_j|^2 \, dx \right)$$

$$\leq c \left( 1 + \int_\Omega |u_j|^3 |u_j|^2 \, dx \right)$$

$$\leq c \left( 1 + \left( \int_\Omega |u_j|^2 \, dx \right)^{\frac{4}{3r}} \left( \int_\Omega (|u_j|^r)^{\frac{2r-2}{r-2}} \, dx \right)^{\frac{r-2}{2}} \right)$$

$$\leq c \left( 1 + \left( \int_\Omega (|u_j|^r)^{\frac{2r-2}{r-2}} \, dx \right)^{\frac{r-2}{2}} \right)$$

$$\leq c \max \left\{ 1, \left( \int_\Omega (|u_j|^r)^{\frac{2r-2}{r-2}} \, dx \right)^{\frac{r-2}{2}} \right\}. \quad (3.5)$$

Let $T \to \infty$ such that $u_j T(x) \to u_j(x)$. By (3.4) and (3.5), we obtain

$$\left( \int_\Omega (|u_j|^r)^{\frac{2r-2}{r-2}} \, dx \right)^{\frac{r}{2}} \leq \frac{cr^2}{2r - 1} \max \left\{ 1, \left( \int_\Omega (|u_j|^r)^{\frac{2r-2}{r-2}} \, dx \right)^{\frac{r-2}{2}} \right\}. \quad (3.6)$$

Denote $d = \frac{2r-2}{2r-1} > 1$, $q = \frac{2r-2}{2r-1} N$. By (3.6), we have

$$\left( \int_\Omega (|u_j|^r)^{qd} \, dx \right)^{\frac{1}{qd}} \leq \left( \frac{cr^2}{2r - 1} \right)^{\frac{1}{qd}} \max \left\{ 1, \left( \int_\Omega (|u_j|^r)^{qd} \, dx \right)^{\frac{1}{qd}} \right\}. \quad (3.7)$$

Choose $r_0$, such that $r_0 d = r_1$ in (3.8), then

$$\left( \int_\Omega |u_j|^r \, dx \right)^{\frac{1}{qd}} \leq \left( \frac{cr_0^2}{2r_0 - 1} \right)^{\frac{1}{qd}} \max \left\{ 1, \left( \int_\Omega |u_j|^r \, dx \right)^{\frac{1}{qd}} \right\}. \quad (3.9)$$

Therefore, by (3.9), we obtain

$$\left( \int_\Omega |u_j|^{r_{k+1}} \, dx \right)^{\frac{1}{r_{k+1}}} \leq \left( \frac{cr_0^2}{2r_0 - 1} \right)^{\frac{1}{r_{k+1}}} \max \left\{ 1, \left( \int_\Omega |u_j|^{r_k} \, dx \right)^{\frac{1}{r_k}} \right\}$$

$$\leq \prod_{i=0}^{k} \left( \frac{cr_i^2}{2r_i - 1} \right)^{\frac{1}{r_{i+1}}} \max \left\{ 1, \left( \int_\Omega |u_j|^{r_0} \, dx \right)^{\frac{1}{r_0}} \right\},$$

where $r_i = d^i r_0$, we denote $C_0 = \prod_{i=0}^{k} \left( \frac{cr_i^2}{2r_i - 1} \right)^{\frac{1}{r_{i+1}}}$, then

$$\|u_j\|_{L^{r_0+k+1}(\Omega)} \leq C_0 (1 + \|u_j\|_{L^{r_2}(\Omega)}). \quad (3.10)$$

Let $k \to \infty$ in (3.10), by (3.1), we have

$$\|u_j\|_{L^\infty(\Omega)} \leq C_0 (1 + \|u_j\|_{L^{r_2}(\Omega)}) \leq c = c(L). \quad \square$$
Proof of Theorem 1.2. By Lemmas 2.2, 2.5–2.8, for a sufficiently small parameter \( \varepsilon \), the functional \( I_M \) satisfies the conditions (A3), (A4)–(A7) of the abstract critical point theorem (Theorem 2.1). Then, \( c_j(M, \varepsilon) \) is a critical value of the functional \( I_M \), and each component of the corresponding critical point \( U_j(M, \varepsilon) \) is sign-changing.

That is, \( U_j(M, \varepsilon) \) is a sign-changing solution of the truncated system (1.3). Moreover, given \( l \in \mathbb{N}^+ \), \( L^* > 0 \), by Lemma 2.8, there exists \( \varepsilon^*_M > 0 \) such that for \( |\varepsilon| \leq \varepsilon^*_M = \min\{\varepsilon^{(1)}_M, \ldots, \varepsilon^{(l)}_M\} \),

\[
c_j(M, \varepsilon) \leq L^* = \max\{d_1, \ldots, d_l\}, \quad j = 1, \ldots, l.
\]

By Lemma 3.1, there exist the constant \( K^* \) independent of \( M \), \( \varepsilon \), and \( \varepsilon_M > 0 \), such that for \( |\varepsilon| \leq \varepsilon_M \),

\[
\|U_j(M, \varepsilon)\|_{L^\infty(\Omega)} \leq K^*, \quad j = 1, \ldots, l.
\]

Now take \( M \geq K^* + 1 \), then for \( |\varepsilon| \leq \varepsilon_l \), \( U_j(\varepsilon) := U_j(M, \varepsilon), j = 1, \ldots, l \) are sign-changing solutions of the perturbed system (1.2).

Note that taking \( \varepsilon = 0 \), we have \( F(U, 0) = 0 \) and \( \frac{\partial F}{\partial u_j}(U, 0) = 0 \), then the solutions to the perturbed system (1.2) are also solutions to the original system (1.1).

In Section 2, we have obtained the sign-changing critical points of the truncated functional \( I_M \). Therefore, by Theorem 1.2, we know that system (1.2) has \( l \) pairs of sign-changing solutions. Then, for \( \varepsilon = 0 \), the system (1.1) has infinitely many sign-changing solutions, and we have thus proved the main result.

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References


**XUE ZHOU**

**DEPARTMENT OF MATHEMATICS, YUNNAN NORMAL UNIVERSITY, KUNMING 650221, CHINA**

*Email address: niuzhoux@163.com*

**XIANGQING LIU (CORRESPONDING AUTHOR)**

**DEPARTMENT OF MATHEMATICS, YUNNAN NORMAL UNIVERSITY, KUNMING 650221, CHINA**

*Email address: lxq8u8@163.com*