MASSERA TYPE THEOREMS FOR ABSTRACT NON-AUTONOMOUS EVOLUTION EQUATIONS

LAN-LING ZHENG, HUI-SHENG DING

Abstract. We establish two fixed point theorems for affine maps in Banach spaces, with weaker assumptions than those in the literature. Then we establish some Massera type results for abstract linear evolution equations without assuming the existence of bounded solutions, which is an indispensable condition in the classical Massera theorem and in the earlier literature. As application, we present an existence result on periodic mild solutions to abstract nonautonomous semilinear evolution equations.

1. Introduction

Let \( X \) be a Banach space. Our aim is to investigate the existence of periodic mild solutions to the nonautonomous evolution equations

\[
    u'(t) = A(t)u(t) + g(t), \quad t \geq 0,
\]

and

\[
    u'(t) = A(t)u(t) + f(t, u(t)), \quad t \geq 0
\]
on \( X \). \( g \) and \( f \) satisfy conditions specified later, and \( \{A(t)\}_{t \geq 0} \) satisfies the following assumptions:

(A1) \( D(A(t)) = D \subset X \) for all \( t \geq 0 \) and \( A(t) \) is not necessarily densely defined, i.e., \( D = X \) is not necessarily true;

(A2) there exist \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( (\omega, +\infty) \subset \rho(A(t)) \) for every \( t \geq 0 \) and

\[
    \left\| \prod_{j=1}^{k}(\lambda I - A(t_j))^{-1} \right\| \leq \frac{M}{(\lambda - \omega)^k}
\]

for every \( \lambda > \omega \) and every finite sequence \( \{t_j\}_{j=1}^{k} \) with \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \), where \( k = 1, 2, \ldots \);

(A3) the mapping \( t \to A(t)x \) is continuously differentiable in \( X \) for every \( x \in D \);

(A4) \( A(t + 1) = A(t) \) for every \( t \geq 0 \).

The existence of periodic solutions to (1.1), (1.2) and their variants has been of great interest for many authors (cf. [1, 3, 4, 5, 7, 9, 8, 11, 12, 13, 14, 18, 19]). To establish the existence of periodic solutions to (1.2), one of the key steps is to consider first the existence of periodic solutions to the linear equation (1.1). It is

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well-known that to find an initial value $x_0 \in X$ for a 1-periodic mild solution of (1.1), one only needs to solve the equation

$$x_0 = P(x_0),$$

where

$$P : X \to X; x \mapsto U(1, 0)x + \int_0^1 U(1, \tau)g(\tau)d\tau$$

stands for the Poincaré mapping and $\{U(t, s)\}_{t \geq s \geq 0}$ is the evolution system generated by $\{A(t)\}_{t \geq 0}$. Now the problem is transformed into a fixed point problem of the Poincaré mapping, an affine map.

For fixed point theorems of Poincaré mapping, one of the celebrated result is by Chow and Hale [2] who proved that $P$ has a fixed point if the range $R(I - U(1, 0))$ is closed and there exists $z_0 \in X$ such that $\sup_{n \in \mathbb{N}} \|P^n z_0\| < +\infty$. Recently, Zubelevich [20] made an important progress and got the result that $P$ has a fixed point if $X$ is reflexive and $\sup_{n \in \mathbb{N}} \|P^n z_0\| < +\infty$ for some $z_0 \in X$. So, Zubelevich removed the closedness of $R(I - U(1, 0))$ in the case of $X$ being reflexive. Very recently, Ezzinbi and Taoudi [6] also established several interesting results in this direction on locally convex spaces and ordered Banach spaces. It is needed to note that the boundedness of $\{P^n z_0\}$ is a key assumption in all the above literature.

As one will see, in this paper, this key assumption is weakened, i.e., we establish two fixed point theorems for Poincaré mapping, where $\{P^n z_0\}$ is not necessarily bounded for some $z_0 \in X$. Moreover, we give two examples, which satisfy our weakened assumptions but not the boundedness condition in the earlier literature [2, 20] (see Section 2).

Based on our new fixed point theorems for Poincaré mapping, we discuss the existence of periodic solutions to (1.1) and obtain two Massera type theorems for (1.1). It is interesting to note that in our Massera type results, the existence of bounded solutions is not presupposed as in the classical Massera theorem. In fact, to the best of our knowledge, of all the latest Massera type results up until now, the existence of bounded solutions is an indispensable assumption (see, e.g., [6, 7, 9, 10] for some recent Massera type results). Moreover, compared with other type results on the existence of periodic solutions for (1.1), our Massera type theorems also have some improvements to some extent. For example, our Massera type theorems do not need the assumptions $\omega < 0$ and $Me^\omega < 1$ in [15]. As application of our Massera type theorems, in the last part of this paper, we establish an existence result on periodic mild solutions to (1.2).

Throughout this paper, we denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of non-negative real numbers, by $\mathbb{C}$ the set of complex numbers, by $\mathbb{N}$ the set of positive integers, by $L^p([a, b], X)$ ($L^p_{loc}(\mathbb{R}^+, X)$) the space of all (locally) pth integrable functions, by $C(\mathbb{R}^+, X)$ the space of all continuous functions from $\mathbb{R}^+$ to $X$, and $P_1(\mathbb{R}^+, X)$ the space of all 1-periodic functions from $\mathbb{R}^+$ to $X$.

2. TWO FIXED POINT THEOREMS FOR AFFINE MAPS

We first present a fixed point theorem, where the boundedness assumption in the classical results by Chow and Hale is weakened.

**Theorem 2.1.** Let $X$ be a Banach space, $B : X \to X$ be a bounded linear operator, $z \in X$ and $P : X \to X; x \mapsto Bx + z$. Assume that the range $R(I - B)$ is closed.
and there exists \( x_0 \in X \) such that
\[
\lim_{n \to +\infty} \frac{P^n x_0}{n} = 0.
\] (2.1)

Then \( P \) has a fixed point \( x \in X \).

**Proof.** Our proof starts with the observation that \( P \) has a fixed point if \( z \in \mathcal{R}(I - B) \). Let \( x_n = \frac{1}{n} \sum_{k=1}^{n} P^k x_0 \) for every \( n \in \mathbb{N} \), then by a direct calculation, we have
\[
\lim_{n \to +\infty} (I - B)x_n = \lim_{n \to +\infty} \frac{1}{n} [(I - P)x_n + z] = \lim_{n \to +\infty} \frac{1}{n} [P x_0 - P^{n+1} x_0] + z = z.
\]
Combining this with \( \mathcal{R}(I - B) \) is closed, we conclude that \( z \in \mathcal{R}(I - B) \). \( \square \)

**Remark 2.2.** In the case of \( \mathcal{R}(I - B) = X \), the conclusion holds without assumption (2.1). Moreover, Chow and Hale [2] obtained the same conclusion of Theorem 2.1 under the condition \( \sup_{n \in \mathbb{N}} \| P^n x_0 \| < +\infty \), which implies that (2.1) holds. However, the converse is not necessarily true. In fact, let
\[
B : l^2 \to l^2; (x_1, x_2, \ldots, x_k, \ldots) \mapsto \left( 0, \sqrt{2} x_1, \sqrt{\frac{3}{2}} x_2, \ldots, \sqrt{\frac{k+1}{k}} x_k, \ldots \right),
\]
and
\[
P : l^2 \to l^2; (x_1, x_2, \ldots, x_k, \ldots) \mapsto B(x_1, x_2, \ldots, x_k, \ldots) + (1, 0, 0, \ldots).
\]
By a direct calculation, we obtain that there exists \( x_0 = (1, 0, 0, \ldots) \in l^2 \) such that
\[
\sup_{n \in \mathbb{N}} \| P^n x_0 \| \geq \sup_{n \in \mathbb{N}} \sqrt{n + 1} = +\infty
\]
and
\[
0 \leq \lim_{n \to +\infty} \frac{P^n x_0}{n} \leq \lim_{n \to +\infty} \frac{(n + 1)^{\frac{3}{2}}}{n} = 0.
\]

If \( X \) have some special properties, then the condition that \( \mathcal{R}(I - B) \) is closed can be removed.

**Theorem 2.3.** Let \( X \) be a Banach space, \( B : X \to X \) be a bounded linear operator, \( z \in X \) and \( P : X \to X; x \to Bx + z \). Assume that there exists \( x_0 \in X \) such that
\[
\sup_{n \in \mathbb{N}} \left\| \frac{1}{n} \sum_{k=1}^{n} P^k x_0 \right\| < +\infty, \quad \lim_{n \to +\infty} \left\| \frac{P^n x_0}{n} \right\| = \frac{1}{n} \sum_{k=1}^{n} P^k x_0 \right\| = 0,
\] (2.2)
and one of the following conditions holds:

(i) \( X \) is reflexive;

(ii) there exists a separable Banach space \( Y \) such that \( X \) is the dual space of \( Y \) and \( B^* Y \subset Y \), where \( B^* \) is the dual operator of \( B \) and \( Y \) is considered as a subspace of \( Y^{**} \).

Then \( P \) has a fixed point \( x \in X \). Moreover,
\[
\| x \| \leq \sup_{n \in \mathbb{N}} \left\| \frac{1}{n} \sum_{k=1}^{n} P^k x_0 \right\|.
\] (2.3)
Proof. Firstly, we prove case (i). Let \( x_n = \frac{1}{n} \sum_{k=1}^{n} P^k x_0 \) for \( n \in \mathbb{N} \). There exist \( \{ x_{n_j} \} \subset \{ x_n \} \) and \( \mathfrak{v} \in X \) such that \( x_{n_j} \) converges weakly to \( \mathfrak{v} \) by using (2.2) and condition (i). So, for every \( x^* \in X^* \)

\[
\lim_{j \to +\infty} \langle x^*, Px_{n_j} - P \mathfrak{v} \rangle = \lim_{j \to +\infty} \langle x^*, Bx_{n_j} - B \mathfrak{v} \rangle = \lim_{j \to +\infty} \langle B^* x^*, x_{n_j} - \mathfrak{v} \rangle = 0,
\]

which means that \( Px_{n_j} \) converges weakly to \( P \mathfrak{v} \). On the other hand, by (2.2) and a direct calculation, we have

\[
\lim_{j \to +\infty} Px_{n_j} - x_{n_j} = \lim_{j \to +\infty} \frac{1}{n_j} [Px_0 - P^{n_j+1} x_0] = 0,
\]

where \( 0 \) is the zero member of \( X \). Therefore,

\[
0 \leq \lim_{j \to +\infty} \|\langle x^*, Px_{n_j} - \mathfrak{v} \rangle\|
\leq \lim_{j \to +\infty} (\|\langle x^*, Px_{n_j} - x_{n_j} \rangle\| + \|\langle x^*, x_{n_j} - \mathfrak{v} \rangle\|) = 0
\]

for every \( x^* \in X^* \), which means that \( Px_{n_j} \) converges weakly to \( \mathfrak{v} \). From this and \( Px_{n_j} \) converges weakly to \( P \mathfrak{v} \), we obtain \( P \mathfrak{v} = \mathfrak{v} \). In addition, (2.3) is obvious if \( \mathfrak{v} = 0 \). When \( \mathfrak{v} \neq 0 \), we know that there exists \( \mathfrak{v}^* \in X^* \) such that \( \|\mathfrak{v}^*\| = 1 \) and

\[
\langle \mathfrak{v}^*, x \rangle = \|\mathfrak{v}\|
\]

for every \( j \in \mathbb{N} \). Combining this with \( x_{n_j} \) converges weakly to \( \mathfrak{v} \), we obtain

\[
\|\mathfrak{v}\| \leq \lim_{j \to +\infty} \left[ \langle \mathfrak{v}^*, \mathfrak{v} - x_{n_j} \rangle + \sup_{n \in \mathbb{N}} \left\| \frac{1}{n} \sum_{k=1}^{n} P^k x_0 \right\| \right] = \sup_{n \in \mathbb{N}} \left\| \frac{1}{n} \sum_{k=1}^{n} P^k x_0 \right\|.
\]

Now, we prove case (ii). Let \( x_n = \frac{1}{n} \sum_{k=1}^{n} P^k x_0 \) for \( n \in \mathbb{N} \). We conclude from (2.2) and condition (ii) that there exist \( \{ x_{n_j} \} \subset \{ x_n \} \) and \( \mathfrak{v} \in X \) such that \( x_{n_j} \) weak*-convergent to \( \mathfrak{v} \), hence that

\[
\lim_{j \to +\infty} \langle Px_{n_j} - P \mathfrak{v}, y \rangle = \lim_{j \to +\infty} \langle Bx_{n_j} - B \mathfrak{v}, y \rangle = \lim_{j \to +\infty} \langle x_{n_j} - \mathfrak{v}, B^* y \rangle = 0
\]

for every \( y \in Y \), which means that \( Px_{n_j} \) weak*-convergent to \( P \mathfrak{v} \). Furthermore, we obtain \( \{ Px_{n_j} - x_{n_j} \} \) weakly*-convergent to \( 0 \) by (2.3). Then

\[
0 \leq \lim_{j \to +\infty} |\langle Px_{n_j} - \mathfrak{v}, y \rangle|
\leq \lim_{j \to +\infty} \left( |\langle Px_{n_j} - x_{n_j}, y \rangle| + |\langle x_{n_j} - \mathfrak{v}, y \rangle| \right) = 0
\]
for every $y \in Y$, which means that $P_{n}$ weak*-convergent to $\pi$. From this and
$P_{n}$ convergent to $P\pi$, we have $P\pi = \pi$. Moreover, since $x_{n}$ is weak*-convergent to $\pi$,
\[
|\langle \pi, y \rangle| = \lim_{j \to +\infty} |\langle x_{n_{j}}, y \rangle| \leq \limsup_{j \to +\infty} \|x_{n_{j}}\| \|y\| = \limsup_{j \to +\infty} \|x_{n_{j}}\| \leq \sup_{n \in \mathbb{N}} \|x_{n}\|
\]

for every $y \in Y$ with $\|y\| = 1$. Therefore,

\[
\|\pi\| \leq \sup_{n \in \mathbb{N}} \|x_{n}\| = \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} P_{k}x_{0}.
\]

Hence, the conclusion is valid.

Zubelevich [20] obtained the conclusion of the above theorem assuming that
\[
\sup_{n \in \mathbb{N}} \|P_{n}x_{0}\| < +\infty
\]

and $X$ is reflexive. We note that $\sup_{n \in \mathbb{N}} \|P_{n}x_{0}\| < +\infty$ implies

\[
\sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} P_{k}x_{0} < +\infty \text{ and } \lim_{n \to +\infty} \frac{P_{n}x_{0}}{n} = 0.
\]

However, the converse is not necessarily true. For example, let

\[
P : l^{2} \to l^{2}; (x_{1}, x_{2}, \ldots, x_{k}, \ldots) \mapsto \left(0, \sqrt{2}x_{1}, \sqrt{3}x_{2}, \ldots, \sqrt{k+1}x_{k}, \ldots \right).
\]

It is easy to see that $P$ is a bounded linear operator and there exists $x_{0} = (1, 0, \ldots) \in l^{2}$ such that

\[
\sup_{n \in \mathbb{N}} \|P_{n}x_{0}\| = \sup_{n \in \mathbb{N}} \sqrt{n+1} = +\infty,
\]

\[
\lim_{n \to +\infty} \frac{P_{n}x_{0}}{n} = \lim_{n \to +\infty} \frac{\sqrt{n+1}}{n} = 0,
\]

\[
\sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} P_{k}x_{0} \leq \sup_{n \in \mathbb{N}} \frac{(n+1)\frac{3}{2}}{n} \leq 2.
\]

3. Massera type theorems for linear evolution equations

In this section, we consider the existence of 1-periodic mild solutions to (1.1)
with the help of our new fixed point theorems for affine maps. Throughout this
section, we assume that $g \in P_{1}(\mathbb{R}^{+}, X) \cap L_{loc}^{1}(\mathbb{R}^{+}, X)$. Next, we first recall the
following definitions and results which are taken from [15] [17] [16].

**Proposition 3.1 ([17])**. Assume that $\{A(t)\}_{t \geq 0}$ satisfies conditions (A1)–(A3) and
$t_{0} > 0$. If $z \in \mathcal{D}$ and $h \in L^{1}(0, t_{0}], X)$, then the limit

\[
u(t) := \lim_{\lambda \to 0^{+}} \left( U_{\lambda}(t, 0)z + \int_{0}^{\frac{t}{\lambda}} U_{\lambda}(t, r)h(r)dr \right).
\]

exists uniformly for $t \in [0, t_{0}]$, and $\nu$ is a continuous function on $[0, t_{0}]$, where

\[
U_{\lambda}(t, 0) = \prod_{j=1}^{\frac{t}{\lambda}} (I - \lambda A(j \lambda))^{-1}
\]

for every $t \in [0, t_{0}]$ and $\lambda > 0$.

The following theorem can be derived from (A1)–(A4) and Proposition 3.1 (see
also [15] [16]).
Proposition 3.2. Assume that \( \{A(t)\}_{t \geq 0} \) satisfies conditions (A1)–(A3). Then the limit
\[
U(t, s)z = \lim_{\lambda \to 0^+} U_{\lambda}(t, s)z
\]
eexists for every \( z \in \mathcal{D} \) and \( (t, s) \in \triangle \), where
\[
U_{\lambda}(t, s) = \prod_{j=\lceil \frac{s}{\lambda} \rceil + 1}^{\lceil \frac{t}{\lambda} \rceil} (I - \lambda A(j\lambda))^{-1}
\]
and \( \triangle = \{(t, s)|t \geq s \geq 0\} \). Also, the families \( \{U(t, s)\}_{t \geq s \geq 0} \) and \( \{U_{\lambda}(t, s)\}_{t \geq s \geq 0} \) satisfy the following properties:
(i) \( U_{\lambda}(t, t)z = z \) and \( U_{\lambda}(t, r)U_{\lambda}(r, s)z = U_{\lambda}(t, s)z \) for every \( z \in \mathcal{D} \), \( \lambda > 0 \) and \( t \geq r \geq s \geq 0 \);
(ii) for every \( \lambda > 0 \) and \( (t, s) \in \triangle \),
\[
\|U_{\lambda}(t, s)\| \leq M \left( \frac{1}{1 - \lambda w} \right)^{\frac{t-s}{\lambda}};
\]
(iii) \( U(t, s) : \mathcal{D} \to \mathcal{D} \) is a bounded linear operator for every \( (t, s) \in \triangle \);
(iv) \( U(t, t)z = z \) and \( U(t, r)U(r, s)z = U(t, s)z \) for every \( z \in \mathcal{D} \) and \( t \geq r \geq s \geq 0 \);
(v) for every \( z \in \mathcal{D} \) and \( (t, s) \in \triangle \),
\[
\|U(t, s)z\| \leq Me^{\omega(t-s)}\|z\|;
\]
(vi) for every \( h \in L_{1\text{loc}}^1(\mathbb{R}^+, X) \) and \( (t, s) \in \triangle \),
\[
U(t, s) \lim_{\lambda \to 0^+} \int_0^s U_{\lambda}(s, r)h(r)dr = \lim_{\lambda \to 0^+} \int_0^s U_{\lambda}(t, r)h(r)dr.
\]
(vii) for every \( h \in L_{1\text{loc}}^1(\mathbb{R}^+, X) \) and \( t \geq 0 \),
\[
\lim_{\lambda \to 0^+} \int_0^{\frac{t}{\lambda}} U_{\lambda}(t, r)h(r)dr = \lim_{\lambda \to 0^+} \int_0^t U_{\lambda}(t, r)h(r)dr;
\]
In addition, if \( \{A(t)\}_{t \geq 0} \) satisfies the condition (A4), then
(viii) for every \( z \in \mathcal{D} \), \( k \in \mathbb{N} \) and \( (t, s) \in \triangle \),
\[
U_{\frac{1}{k}}(t + 1, s + 1)z = U_{\frac{1}{k}}(t, s)z;
\]
(ix) for every \( z \in \mathcal{D} \) and \( (t, s) \in \triangle \),
\[
U(t + 1, s + 1)z = U(t, s)z.
\]

Definition 3.3 (12). A continuous function \( u : [0, +\infty) \to \mathcal{D} \) is called a mild solution of (1.2) if \( u \) satisfies
\[
u(t) = U(t, s)u(s) + \lim_{\lambda \to 0^+} \int_t^s U_{\lambda}(t, r)f(r, u(r))dr
\]
for every \( (t, s) \in \triangle \). Then, a continuous function \( u : [0, +\infty) \to \mathcal{D} \) is called a mild solution of (1.1) if \( u \) satisfies
\[
u(t) = U(t, s)u(s) + \lim_{\lambda \to 0^+} \int_t^s U_{\lambda}(t, r)g(r)dr
\]
for every \( (t, s) \in \triangle \).
Remark 3.4. Let \( g \in L^1_{loc}(\mathbb{R}^+, X) \). By Proposition 3.1 and (3.5), we note that \( u \) is continuous if \( u \) satisfies (3.8).

Remark 3.5. Let \( t_0 > 0 \). According to [17, Theorem 4.2], if \( g \in W^{1,1}([0, t_0], X) \), \( x \in D \), and \( A(0)x + g(0) \in \overline{D} \), then

\[
\lim_{t \to t_0} u(t) = u(t_0).
\]

By using (3.4), Remark 3.4 and (iv) in Proposition 3.1, we have

\[
u : [0, t_0] \to \overline{D}; t \mapsto U(t, 0)x + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t, r)g(r)dr
\]
is a classical solution to (1.1) on \([0, t_0] \), where

\[
W^{1,1}([0, t_0], X) = \{ u \in L^1([0, t_0], X) : u(t) = u_0 + \int_0^t v(s)ds \text{ for some } u_0 \in X \text{ and } v \in L^1([0, t_0], X), t \in [0, t_0] \}.
\]

Using Theorems 2.1 and 2.3, we obtain the following results.

Theorem 3.6. Assume that there exists a mild solution \( u_0 \) of (1.1) such that

\[
\lim_{n \to +\infty} \| u_0(n) \|_n = 0
\]
and \( R(I - U(1, 0)) \) is closed, where \( I \) is the identity map and \( R(I - U(1, 0)) \) is the range of \( I - U(1, 0) \). Then (1.1) has a 1-periodic mild solution \( u \) satisfying

\[
\|u\| \leq Me^{|z|} \left( \|u(0)\| + \int_0^1 \|g(\sigma)\|d\sigma \right)
\]

Furthermore, the 1-periodic mild solution of (1.1) is unique if

\[
\lim_{t \to +\infty} \|U(t, 0)x\| = 0 \quad \text{for every } x \in \overline{D} \text{ with } \sup_{t \geq 0} \|U(t, 0)x\| < +\infty.
\]

Proof. Let

\[
P : \overline{D} \to \overline{D}; x \mapsto U(1, 0)x + \lim_{\lambda \to 0^+} \int_0^1 U_\lambda(1, r)g(r)dr.
\]

Take \( X = \overline{D} \), \( B = U(1, 0) \), \( z = \lim_{\lambda \to 0^+} \int_0^1 U_\lambda(1, r)g(r)dr \), and \( x_0 = u_0(0) \) in Theorem 2.1. Note that \( P^k(x_0) = u_0(k) \) for every \( k \in \mathbb{N} \). By Theorem 2.1 \( P \) has a fixed point \( \overline{x} \in \overline{D} \). Let

\[
u : [0, +\infty) \to X; t \mapsto U(t, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t, r)g(r)dr
\]

By using (3.4), Remark 3.4 and (iv) in Proposition 3.1, we have

\[
U(t, s)u(s) + \lim_{\lambda \to 0^+} \int_s^t U_\lambda(t, r)g(r)dr
\]

\[
= U(t, s) \left[ U(s, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^s U_\lambda(s, r)g(r)dr \right] + \lim_{\lambda \to 0^+} \int_s^t U_\lambda(t, r)g(r)dr
\]

\[
= U(t, 0)\overline{x} + U(t, s) \lim_{\lambda \to 0^+} \int_0^s U_\lambda(s, r)g(r)dr + \lim_{\lambda \to 0^+} \int_s^t U_\lambda(t, r)g(r)dr
\]

\[
= U(t, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^s U_\lambda(t, r)g(r)dr + \lim_{\lambda \to 0^+} \int_s^t U_\lambda(t, r)g(r)dr
\]

\[
= U(t, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t, r)g(r)dr = u(t)
\]
for every \((t, s) \in \Delta\), which means that \(u\) is a mild solution to (1.1). Moreover, by \(g \in P_1(\mathbb{R}^+, X)\), (3.4), (3.6), (3.7) and (iv) in Proposition 3.2, we have

\[
\begin{align*}
    u(t) &= U(t, 0)\varphi + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t, r)g(r)dr \\
    &= U(t, 0)P\varphi + \lim_{k \to +\infty} \int_0^t U_\frac{1}{k}(t, r)g(r)dr \\
    &= U(t, 0)\left[U(1, 0)\varphi + \lim_{\lambda \to 0^+} \int_0^1 U_\lambda(1, r)g(r)dr\right] \\
    &\quad + \lim_{k \to +\infty} \int_0^t U_\frac{1}{k}(t + 1, r + 1)g(r + 1)dr
    = U(t + 1, 1)\left[U(1, 0)\varphi + \lim_{\lambda \to 0^+} \int_0^1 U_\lambda(1, r)g(r)dr\right] \\
    &\quad + \lim_{k \to +\infty} \int_1^{t+1} U_\frac{1}{k}(t + 1, r)g(r)dr \\
    &= U(t + 1, 0)\varphi + \lim_{\lambda \to 0^+} \int_0^1 U_\lambda(t + 1, r)g(r)dr + \lim_{\lambda \to 0^+} \int_1^{t+1} U_\lambda(t + 1, \tau)g(\tau) d\tau \\
    &= U(t + 1, 0)\varphi + \int_0^{t+1} U_\lambda(t + 1, r)g(r)dr = u(t + 1)
\end{align*}
\]

for every \(t \geq 0\), which means that \(u\) is \(1\)-periodic. In addition, let \(u_1\) and \(u_2\) be two \(1\)-periodic mild solutions to (1.1). Note that \(u_1(0) - u_2(0) \in D\) and

\[
    \sup_{t \geq 0} \|U(t, 0)(u_1(0) - u_2(0))\| = \sup_{t \geq 0} \|u_1(t) - u_2(t)\| < +\infty.
\]

So, \(\lim_{t \to +\infty} \|u_1(t) - u_2(t)\| = \lim_{t \to +\infty} \|U(t, 0)(u_1(0) - u_2(0))\| = 0\) by (3.10). Combining this with \(u_1 - u_2 \in P_1(\mathbb{R}^+, X)\), we obtain \(u_1 = u_2\). Thus, the \(1\)-periodic mild solution to (1.1) is unique. Furthermore, we obtain (3.9) by using (3.2) and (3.3).

**Theorem 3.7.** Assume that there exists a mild solution \(u_0\) of (1.1) such that

\[
    \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^n u_0(k) < +\infty, \quad \lim_{n \to +\infty} \frac{u_0(n)}{n} = 0,
\]

and one of the following conditions holds:

(i) \(X\) is reflexive;

(ii) there exists a separable Banach space \(Y\) such that \(D\) is the dual space of \(Y\) and \(U^*(1, 0)Y \subset Y\), where \(U^*(1, 0)\) is the dual operator of \(U(1, 0)\) and \(Y\) is considered as a subspace of \(Y^{**}\).

Then (1.1) has a \(1\)-periodic mild solution \(u\) satisfying

\[
    \|u\| \leq Me^{\omega\tau} \left(\sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^n u_0(k) + \int_0^1 \|g(\sigma)\| d\sigma\right), \quad (3.12)
\]

Furthermore, the \(1\)-periodic mild solution of (1.1) is unique if (3.10) holds.

**Proof.** By using Theorem 2.3 similar to the proof of Theorem 3.6, one can show that (1.1) has a \(1\)-periodic mild solution \(u\) given by (3.11) and \(u\) satisfies (3.9). In
addition, according to (2.3), we have
\[
\|u(0)\| = \|\mathcal{T}\| \leq \sup_{n \in \mathbb{N}} \left\| \frac{1}{n} \sum_{k=1}^{n} P_k(x_0) \right\| = \sup_{n \in \mathbb{N}} \left\| \frac{1}{n} \sum_{k=1}^{n} u_0(k) \right\|.
\]
Combining this with (3.9), we obtain (3.12). \(\square\)

4. Application to semilinear evolution equations

Next, we establish the existence of 1-periodic mild solutions for (1.2) by our Massera type results obtained in the previous section. For convenience, we list some assumptions:

(A5) \(f(\cdot, x) \in \mathcal{P}_{1}(\mathbb{R}^+, X) \cap L^{1}_{\text{loc}}(\mathbb{R}^+, X)\) for every \(x \in X\);
(A6) there exist \(\varrho > 0\) and \(L > 0\) such that
\[
\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|
\]
for every \(t \geq 0\) and \(x_1, x_2 \in X\) with \(\|x_1\|, \|x_2\| \leq \varrho\).

From now on, we denote \(\gamma = \int_{0}^{1} \|f(\sigma, 0)\| d\sigma\).

**Theorem 4.1.** Assume that
(a) there exists \(\alpha > 0\) with
\[
Me^{\omega\varrho}(\alpha + 1) = \min \left\{ \frac{1}{L}, \frac{\varrho}{\varrho L + \gamma} \right\}
\]
such that (1.1) has a mild solution \(u^{\alpha}_{0}\) satisfies
\[
\sup_{n \in \mathbb{N}} \left\| \frac{1}{n} \sum_{k=1}^{n} u^{\alpha}_{0}(k) \right\| \leq \alpha \int_{0}^{1} \|g(\sigma)\| d\sigma \quad \text{and} \quad \lim_{n \to +\infty} \frac{u^{\alpha}_{0}(n)}{n} = 0 \quad (4.2)
\]
for every \(g \in \mathcal{P}_{1}(\mathbb{R}^+, X) \cap L^{1}_{\text{loc}}(\mathbb{R}^+, X)\);
(b) \(\lim_{t \to +\infty} \\sup_{t \geq 0} \|U(t, 0)x\| = 0\) for every \(x \in \overline{D}\) with \(\sup_{t \geq 0} \|U(t, 0)x\| < +\infty\);
(c) one of the following conditions holds:
(i) \(X\) is reflexive;
(ii) there exists a separable Banach space \(Y\) such that \(\overline{D}\) is the dual space of \(Y\) and \(U^*(1, 0)Y \subset Y\), where \(U^*(1, 0)\) is the dual operator of \(U(1, 0)\) and \(Y\) is considered as a subspace of \(Y^{**}\).

Then (1.2) has a 1-periodic mild solution \(u\).

**Proof.** Let \(B_{\varrho} = \{v \in \mathcal{P}_{1}(\mathbb{R}^+, X) \cap C(\mathbb{R}^+, X) | \sup_{t \in \mathbb{R}^{+}} \|v(t)\| \leq \varrho\}\) and
\[
T : B_{\varrho} \to B_{\varrho}: v \mapsto u^{v},
\]
where \(u^{v}\) is the 1-periodic mild solution of (1.1) with \(g = f(\cdot, v(\cdot))\) by Theorem 3.7.

**Step 1.** We show that \(Tv \in B_{\varrho}\) for every \(v \in B_{\varrho}\). Let \(v \in B_{\varrho}\). Then we know from Theorem 3.7 that (1.1) has a unique 1-periodic mild solution \(u^{v}\) satisfying
\[
\|u^{v}\| \leq Me^{\omega\varrho} \left[ \sup_{n \in \mathbb{N}} \left\| \frac{1}{n} \sum_{k=1}^{n} u^{\alpha}_{0}(k) \right\| + \int_{0}^{1} \|g(\sigma)\| d\sigma \right],
\]
Combining this with (4.2), (A5) and (A6), we obtain
\[
\|u^{v}\| \leq Me^{\omega\varrho}(\alpha + 1) \int_{0}^{1} \|f(\sigma, v(\sigma))\| d\sigma,
\]
\[ T^{v}v \leq Me^{[\omega](\alpha + 1)} \left( L \sup_{t \in \mathbb{R}^+} \| v(t) \| + \int_{0}^{1} \| f(\sigma, 0) \| d\sigma \right) \]
\[ \leq Me^{[\omega](\alpha + 1)}(L\rho + \gamma) \leq \rho. \]

So, \(Tv = u^v \in B_\rho\) by (4.1).

**Step 2.** We show that \(T\) is a contraction. Let \(v_1, v_2 \in B_\rho\). Then
\[ \|Tv_1 - Tv_2\| = \sup_{t \in [0,1]} \|u^{v_1}(t) - u^{v_2}(t)\| \]
\[ = \sup_{t \in [0,1]} \|U(t, 0)(u^{v_1}(0) - u^{v_2}(0)) + \lim_{\lambda \to 0^+} \int_{0}^{t} U_\lambda(t, r)(f(r, v_1(r)) - (f(r, v_2(r)))) dr\| \]
\[ \leq Me^{[\omega](\alpha + 1)}L \sup_{t \in \mathbb{R}^+} \| v_1(t) - v_2(t) \| \]
\[ \leq Me^{[\omega](\alpha + 1)}L \|v_1 - v_2\|, \]
where \(u^{v_1}\) and \(u^{v_2}\) are the corresponding 1-periodic mild solution to (1.1) with \(g_1 = f(\cdot, v_1(\cdot))\) and \(g_2 = f(\cdot, v_2(\cdot))\), respectively. By (4.1), we conclude that \(T\) is a contraction. So \(T\) has a unique fixed point \(u \in B_\rho\), i.e.,
\[ u(t) = U(t, s)u(s) + \lim_{\lambda \to 0^+} \int_{s}^{t} U_\lambda(t, \tau)f(\tau, u(\tau)) d\tau, \quad t \geq s \geq 0, \]
which is a 1-periodic mild solution for (1.2).

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**References**


Lan-Ling Zheng
School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi 330022, China
E-mail address: 202150000054@jxnu.edu.cn

Hui-Sheng Ding (corresponding author)
School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi 330022, China
E-mail address: dinghs@mail.ustc.edu.cn