EXISTENCE OF GLOBAL WEAK SOLUTIONS FOR A 
\( p \)-LAPLACIAN INEQUALITY WITH STRONG DISSIPATION IN 
NONCYLINDRICAL DOMAINS

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Abstract. In this work, we obtain global solutions for nonlinear inequalities 
of \( p \)-Laplacian type in noncylindrical domains, for the unilateral problem with 
strong dissipation
\[
u'' - \Delta_p u - \Delta u' - f \geq 0 \quad \text{in } Q_0,
\]
where \( \Delta_p \) is the nonlinear \( p \)-Laplacian operator with \( 2 \leq p < \infty \), and 
\( Q_0 \) is the noncylindrical domain. Our proof is based on a penalty argument by J. L. 
Lions and Faedo-Galerkin approximations.

1. Introduction

Let \( \Omega \subset \mathbb{R}^n \) be an open and bounded set with smooth boundary \( \Gamma \), \( T \) be a 
positive real, fixed but arbitrary, and \( Q_0 = \Omega \times (0, T) \) be the cylinder with side 
border \( \Sigma_0 = \Gamma_0 \times (0, T) \). J. L. Lions \cite{Lions} considered the problem
\[
u'' - \Delta u - f \geq 0 \quad \text{in } Q_0, \\
u' \geq 0 \quad \text{in } Q_0, \\
u = 0 \quad \text{on } \Sigma_0, \\
u(0) = u_0, \quad u'(0) = u_1 \quad \text{in } \Omega.
\]

If \( K = \{ v \in H_0^1(\Omega); v(x) \geq 0 \text{ a.e. on } \Omega \} \), then \( \text{(1.1)} \) can be reformulated as
\[
\langle u''(t), v - u'(t) \rangle + \langle -\Delta u(t), v - u'(t) \rangle \geq \langle f(t), v - u'(t) \rangle, \quad \forall v \in K,
\]
\[
u'(t) \in K \quad \text{a.e.},
\]
\[
u(0) = u_0, \quad u'(0) = u_1.
\]

We consider the \( p \)-Laplacian operator \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \), which can be 
extended to a monotone, bounded, hemicontinuous and coercive operator between 
the spaces \( W_0^{1,p}(\Omega) \) and its dual by
\[
-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega), \quad \langle -\Delta_p u, v \rangle_p = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx.
\]
The existence of a global solution for the wave equation of \( p \)-Laplacian type

\[
\frac{\partial^2 u}{\partial t^2} - \Delta_p u = 0 \tag{1.3}
\]

without an additional dissipation term, is an open problem. For \( n = 1 \), Derher [5] gave the finite time existence of solution and showed, by a generic counter-example, that the global solution not can be expected. Later, adding a strong dissipation \((-\Delta u')\) in (1.3), the well-posedness and asymptotic behavior it was studied by Greenberg [9]. Nevertheless, when the strong damping is replaced by weaker damping \((\frac{\partial u}{\partial t})\), existence and uniqueness of a global solution are only known for \( n = 1, 2 \). See [4, 22]. Gao and Ma [8] proved global existence of solution and asymptotic behavior under the intermediate damping \((-\Delta)\alpha u'\) with \( 0 < \alpha \leq 1 \).

The memory damping was analyzed by Raposo et al. [18], \( p \)-Laplacian damping was studied by Pereira et al. [16] and a thermoelastic effect was considered in [20]. For wave coupled systems of the \( p \)-laplacian type see [15]. For other works on this subject, we cite [14] and the references therein. For a brief review of the literature on non-cylindrical domain, we cite [1, 7, 11, 17]. Unilateral problem is very interesting because in general, dynamic contact problems are characterized by nonlinear hyperbolic variational inequalities. For contact problems in elasticity and finite element method see Kikuchi-Oden [10] and reference therein. For contact problem viscoelastic materials, see Rivera and Oquendo [21]. For dynamic contact problems with friction, for instance, problems involving unilateral contact with dry friction of Coulomb, see Ballard and Basseville [3]. The study of variational inequalities in bounded domains has been analyzed by several authors, for example, see [2, 6, 12, 13].

In this work we consider the following \( p \)-Laplacian unilateral problem with strong dissipation,

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \Delta_p u - \Delta u' - f &\geq 0 \quad \text{in } Q_0, \\
u' &\geq 0 \quad \text{in } Q_0, \\
u &\geq 0 \quad \text{on } \Sigma_0, \\
u(0) = u_0, \quad u'(0) = u_1 \quad \text{in } \Omega.
\end{align*}
\tag{1.4}
\]

We prove the existence of solutions for (1.4) by using the penalty method.

2. Penalty method

When using the penalization technique as in [11], a difficulty may appear since the term \( \langle u''(t), v - u'(t) \rangle \) makes sense only when \( u''(t) \in H^{-1}(\Omega) \), which is not always possible to obtain. For this reason, the result obtained is the weak formulation of (1.4), namely: if \( K \subset W_0^{1,p}(\mathbb{R}^n) \) is a closed and convex subset with \( 0 \in K \), and

\[
V = \{ v \in L^2(0,T;W_0^{1,p}(\Omega)); v' \in L^2(0,T,W_0^{-1,p'}(\Omega_t)), v(t) \in K \text{ a.e.} \},
\]

\[
K = \{ v \in W_0^{1,p}(\Omega); v(x) \geq 0 \text{ a.e. in } \Omega \},
\]

equation (1.4) can be reformulated as

\[
\langle u''(t), v - u'(t) \rangle + \langle \Delta_p u(t), v - u'(t) \rangle + \langle -\Delta u'(t), v - u'(t) \rangle \geq \langle f(t), v - u'(t) \rangle,
\]

\[
u''(t) \geq f(t), \quad u''(0) = u_0, \quad u'(0) = u_1,
\]

for \( u'(t) \in K \text{ a.e. and for all } v \in K \).
Then, the existence of \( u : Q \rightarrow \mathbb{R} \), with
\[
 u(0) = u_0 \in W_{0}^{1,p}(\Omega_0), \; u'(0) = u_1 \in L^2(\Omega_0) \cap K, \quad u'(t) \in K \text{ a.e. in } (0, T)
\]
and
\[
\int_0^t \langle v'(t) + \Delta_p u(t) - \Delta u'(t) - f(t), v(t) - u'(t) \rangle \geq \frac{1}{2} |v(s) - u'(s)|^2 - \frac{1}{2} |v(0) - u'(0)|^2, \; \forall s \in (0, T), \; \forall v \in V.
\]
(2.2)

It is easy to check that if \( u' \in V \), then (2.1) and (2.2) are equivalent. However, we shall find a solution for (1.4) in the sense of (2.2). Thus, the objective of this work is to obtain the existence of global weak solution to (1.4) considering \( Q \) as a non-cylindrical domain, as in fact, Lions [11] provides existence and uniqueness of weak solutions and/or regular for operators of the parabolic-hyperbolic type in the noncylindrical domain.

By \( D(\Omega) \) we denote the space of infinitely differentiable functions with compact support contained in \( \Omega \). The inner product and norm in \( L^2(\Omega) \) and \( H^1_0(\Omega) \) will be represented by \( \langle \cdot, \cdot \rangle \), \( |\cdot| \), \( ||\cdot|| \), respectively, and by \( \langle \cdot, \cdot \rangle \) the duality between \( W_{0}^{1,p}(\Omega) \) and \( W^{-1,p}(\Omega) \).

If \( T > 0 \) and \( X \) is a Banach space with the norm \( ||\cdot||_X \), we denote by \( L^p(0, T; X) \), \( 1 \leq p < +\infty \), the Banach space of vector functions \( u : (0, T) \rightarrow X \) that are measurable and \( ||u(t)||_X \in L^p(0, T) \) with the norm
\[
||u||_{L^p(0, T; X)} = \left[ \int_0^T ||u(t)||_X^p dt \right]^{1/p},
\]
and by \( L^\infty(0, T; X) \) the Banach space of vector functions \( u : (0, T) \rightarrow X \) that are measurable and \( ||u(t)||_X \in L^\infty(0, T) \) with the norm
\[
||u||_{L^\infty(0, T; X)} = \text{esssup}_{0 < t < T} ||u(t)||_X.
\]

Let \( \Omega \) be an open, connected and bounded subset of \( \mathbb{R}^n \) with regular boundary \( \Gamma, \; Q \subset \Omega_0 \) a noncylindrical domain. We will use the following notation
\[
\Omega_s = Q \cap \{ t = s \} \text{ for } 0 < s < T, \quad \Omega_0 = \text{int}_{\mathbb{R}^n}(\overline{Q} \cap \{ t = 0 \}),
\]
\[
\Omega_T = \text{int}_{\mathbb{R}^n}(\overline{Q} \cap \{ t = T \}), \quad \Gamma_s = \partial \Omega_s,
\]
\[
\Sigma = \cup_{0 < s < T} \Gamma_s, \quad \partial Q = \Omega_0 \cup \Sigma \cup \Omega_T \quad \text{the boundary of } Q.
\]

It is clear that \( \Omega_0 \neq \emptyset \). Our hypotheses on \( Q \) are:
(H1) \( \Omega_t \) is monotonically increasing, that is, \( \Omega_t^* \subset \Omega_s^* \) if \( t < s \), where \( \Omega_t^* = \text{Proj}_{\{t=0\}} \Omega_t \).

(H2) For each \( t \in [0, T] \), \( \Omega_t \) has the following regularity property: if \( u \in W_{0}^{1,p}(\Omega^*_t) \) and \( u = 0 \text{ a.e. in } (\Omega \setminus \Omega_t^*) \), then \( u|_{\Omega_t^*} \in H^1_0(\Omega_t^*) \).

To simplify the notation, we identify \( \Omega_t^* \) with \( \Omega_t \).

Let us define
\[
L^q(0, T; L^p(\Omega_t)) = \{ w \in L^q(0, T; L^p(\Omega_t)) : w = 0 \text{ a.e. in } Q_0 \setminus Q \}.
\]
When \( 1 \leq q < \infty \) we consider the norm
\[
||w||_{L^p(0, T; L^p(\Omega_t))] = \left[ \int_0^T ||w(t)||_{L^p(\Omega_t)}^q dt \right]^{1/q},
\]
which coincides with \(\|w\|_{L^q(0,T;L^p(\Omega))}\). And when \(q = \infty\) we consider
\[
\|w\|_{L^\infty(0,T;L^p(\Omega))} = \sup_{0 < t < T} \|w(t)\|_{L^p(\Omega)}.
\]

Note that \(L^q(0,T;L^p(\Omega_t))\) is a closed subspace of \(L^q(0,T;L^p(\Omega))\) for \(1 \leq q \leq \infty\). Analogously we define \(L^q(0,T;W^{1,p}_0(\Omega_t))\), \(1 \leq q \leq \infty\). It is also true that \(L^q(0,T;W^{1,p}_0(\Omega_t))\) is a closed subspace of \(L^q(0,T;W^{1,p}_0(\Omega))\).

3. Existence of a global weak solution

**Theorem 3.1.** Let \(f \in L^2(0,T;L^2(\Omega))\), \(u_0 \in W^{1,p}_0(\Omega_0)\), \(u_1 \in L^2(\Omega_0) \cap K\), with \(K\) being a convex and closed subset of \(W^{1,p}_0(\Omega)\), and \(0 \in K\). Let us suppose that (H1) and (H2) are satisfied. Then there exists a function \(u : Q \to \mathbb{R}\) satisfying
\[
u \in L^\infty(0,T;W^{1,p}_0(\Omega_t)),
\]
\[
u' \in L^\infty(0,T;L^2(\Omega_t)) \cap L^2(0,T;H^1_0(\Omega_t)),
\]
\[
u'(t) \in K \, \text{a.e. in } (0,T),
\]
\[
u(t) \to \tilde{u}_0 \text{ in } H^1_0(\Omega) \text{ if } t \to 0,
\]
\[
u'(t) \to \tilde{u}_1 \text{ in } L^2(\Omega) \text{ if } t \to 0,
\]
\[
\int_0^s \langle \nu'(t) + \Delta_p u(t) - \Delta u(t) - f(t), v(t) - \nu'(t) \rangle dt \geq \tfrac{1}{2} |v(s) - \nu'(s)|^2_{L^2(\Omega_t)} - \tfrac{1}{2} |v(0) - v(1)|_{L^2(\Omega_0)}, \quad \forall s \in (0,T), \forall v \in V,
\]
\[
u(0) = u_0, \quad \nu'(0) = u_1 \text{ in } \Omega_0.
\]
with \(\tilde{u}_0\) and \(\tilde{u}_1\) being extensions of \(u_0\) and \(u_1\) to \(\Omega\) that vanish outside \(\Omega_0\).

**Theorem 3.1** will be proved by using the Faedo-Galerking method, penalty operator associated to the convex set and penalty method from Lions [11]. At First we find a solution of penalized problem in the cylinder \(Q_0\) and then we show that the restriction to the noncylindrical domain \(Q\) is indeed weak solution for the original problem.

To this end, let \(\tilde{u}_0 \in W^{1,p}_0(\Omega)\), \(\tilde{u}_1 \in L^2(\Omega)\), and \(\tilde{f} \in L^2(Q_0)\) be the extensions to zero outside \(\Omega_0\) of \(u_0\), \(u_1\), and \(f\), respectively. Let us also consider a penalty function for noncylindrical domains:
\[
M(x,t) = \begin{cases} 0 & \text{in } Q \cup \Omega_0 \times \{0\}, \\ 1 & \text{in } Q_0 \setminus (Q \cup \Omega_0 \times \{0\}). \end{cases}
\]

Let \(P_K : H^1_0(\Omega) \to K\) be the projection operator: for \(u \in H^1_0(\Omega)\), \(P_K u\) is the unique element in \(K\) such that
\[
\|u - P_K u\| \leq \|u - k\|, \quad \forall k \in K,
\]
where \(r\) is a fixed integer with \(r > 1 + \frac{p}{2} - \frac{n}{p}\) such that \(H^r_0(\Omega) \hookrightarrow W^{1,p}_0(\Omega)\) continuously. Let \(J\) be the duality operator from \(H^r_0(\Omega)\) into \(H^{-r}(\Omega)\) relatively to the identity from \(R_+\) to \(R_+\). That is,
\[
(Ju, u) = \|Ju\|_{H^{-r}(\Omega)} \|u\|,
\]
\[
\|J(u)\|_{H^{-r}(\Omega)} = \|u\|.
\]
We consider now $\beta : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by $\beta(u) = J(u - P_Ku)$. The operator $\beta$ is a penalty operator associated to $K$, thus satisfies

$$\beta \text{ is monotone, bounded, Hemicontinuous and}$$

$$K = \{ v \in H_0^1(\Omega); \beta(v) = 0 \}. \tag{3.8}$$

The proof of Theorem 3.1 is a consequence of the following theorem.

**Theorem 3.2.** Suppose the hypotheses of Theorem 3.1 are satisfied. Then for each $\mu > 0$ there exists a function $u_\mu : Q_0 \rightarrow \mathbb{R}$ satisfying

$$u_\mu \in L^\infty(0, T; W^{1,p}_0(\Omega)), \tag{3.9}$$

$$u_\mu' \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \tag{3.10}$$

$$\int_0^s \left[ (v'(t) + \Delta_p u_\mu(t) - \Delta u_\mu'(t) - f(t), v(t) - u_\mu'(t) + \frac{1}{\mu} \langle M(t) u_\mu'(t), v(t) \rangle) \right] dt \geq \frac{1}{2} |v(s) - u_\mu(s)|^2 - \frac{1}{2} |v(0) - u_1|^2, \quad \forall t \in (0, T), \tag{3.11}$$

$$\forall \mu, \forall v \in L^2(0, T; W^{1,p}_0(\Omega)) \text{ such that } v' \in L^2(0, T, W^{-1,p'}(\Omega)).$$

Before prove the main theorem, we present the existence of a special basis.

### 4. Galerkin basis

According [19], we will show that there exists a Hilbert space $H_0^s(\Omega)$ with $0 < s$ such that $H_0^s(\Omega) \hookrightarrow W^s_0(\Omega)$ is continuous and $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is continuous and compact.

For $v \in H^1(\mathbb{R}^n)$ we consider Fourier transform of $v$,

$$\hat{v}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} v(x) dx$$

and

$$H^s(\mathbb{R}^n) = \{ v \in L^2(\mathbb{R}^n) : (1 + |\xi|^{s/2}) \hat{v}(\xi) \in L^2(\mathbb{R}^n) \}.$$  

Since $\Omega$ is a bounded open set with sufficiently smooth boundary, we have $H^s(\Omega)$ is the set of restrictions on $\Omega$ of the functions $v \in H^s(\mathbb{R}^n)$, then

$$\|v\|_{H^s(\Omega)} = \inf \{ \|V\|_{H^s(\mathbb{R}^n)} : V = v \text{ a.e. in } \Omega \}$$

and

$$H_0^s(\Omega) = \overline{C^\infty_0(\Omega)}^{H^s(\Omega)}.$$

We need

$$W^{m,q}_0(\Omega) \hookrightarrow W^{m-k,q_k}_0(\Omega), \quad \frac{1}{q_k} = \frac{1}{q} - \frac{k}{n}.$$  

Choosing $q_k = p$, $m - k = 1$ and $q = 2$ we obtain $m = 1 + \frac{n}{2} - \frac{n}{p}$. For $s > m$ we have

$$H_0^s(\Omega) \hookrightarrow W^{1,p}_0(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega)$$

from where our goal follows. Now, from spectral theory the problem

$$((v_j, v))_{H_0^s(\Omega)} = \lambda_j(v_j, v), \quad \text{for all } v \in H_0^s(\Omega)$$

has solution and moreover $\{v_j\}_{j \in \mathbb{N}}$ precisely, is a Schauder basis for $H_0^s(\Omega) \cap L^{r+1}(\Omega)$ with elements that are orthogonal in $L^2(\Omega)$.  

5. Proof of the main theorem

The proof of Theorem 3.2 will be made in 4 steps.

5.1. Penalty approximated problem. Let \{w_1, w_2, \ldots\} be a Schauder basis of $H^1_0(\Omega)$ as demonstrate before, and for each $m \in \mathbb{N}$ let $V_m = [w_1, \ldots, w_m]$ be the subspace generated by the $m$ first vectors from this basis. Let $0 < \varepsilon < 1$ fixed. We wish to find

$$u_{\varepsilon \mu m}(x, t) := u_{\varepsilon \mu m}(t) = \sum_{j=1}^{m} g_{\varepsilon \mu m}(t) w_j(x),$$

where $g_{\varepsilon \mu m}(t)$ of the system of ODEs

$$\begin{align*}
(u''_{\varepsilon \mu m}(t), w_j) + (\Delta_p u_{\varepsilon \mu m}(t), w_j) + (\nabla u_{\varepsilon \mu m}(t), \nabla w_j) \\
+ \frac{1}{\varepsilon} (\beta(u_{\varepsilon \mu m}(t)), w_j) + \frac{1}{\mu} (M(t) u_{\varepsilon \mu m}(t), w_j) \\
= (f(t), w_j), \quad \forall w_j \in V_m,
\end{align*}
$$

(5.1)

and

$$
\begin{align*}
\int_{0}^{t} (\varepsilon \mu m) &= u_{0m} \rightarrow \tilde{u}_0 \quad \text{strongly in } W^{1,p}_0(\Omega), \\
u_{\varepsilon \mu m}'(0) &= u_{1m} \rightarrow \tilde{u}_1 \quad \text{strongly in } L^2(\Omega).
\end{align*}
$$

(5.2)

(5.3)

By Caratheodory the system (5.1) has a local solution $u_{\varepsilon \mu m}(t)$ defined in some interval $[0, t_m)$, $0 < t_m < T$.

5.2. A priori estimates I. Composing (5.1) with $u_{\varepsilon \mu m}'(t) \in V_m$ and then integrating from 0 to $t < t_m$, we obtain

$$
\begin{align*}
&\frac{1}{2} \left\{ |u'_{\varepsilon \mu m}(t)|^2 + \frac{1}{p} \|u_{\varepsilon \mu m}(t)\|_{W^{1,p}_0(\Omega)}^p \right\} + \int_{0}^{t} \|u'_{\varepsilon \mu m}(s)\|^2_{H^1_0(\Omega)} ds \\
+ \frac{1}{\varepsilon} \int_{0}^{t} (\beta(u_{\varepsilon \mu m}'(s)), u_{\varepsilon \mu m}'(s)) ds + \frac{1}{\mu} \int_{0}^{t} (M(t) u_{\varepsilon \mu m}'(s), u_{\varepsilon \mu m}'(s)) ds \\
= \int_{0}^{t} (f(s), u_{\varepsilon \mu m}'(s)) ds + \frac{1}{2} |u_{0m}|^2 + \frac{1}{p} \|u_{1m}\|_{W^{1,p}_0(\Omega)}^p,
\end{align*}
$$

(5.4)

Using (5.2) and (5.3), the monotonicity of $\beta$, the definition of $M$, $\tilde{f} \in L^2(Q_0)$, and Gronwall’s lemma in (5.3), we obtain

$$
\begin{align*}
&\frac{1}{2} |u'_{\varepsilon \mu m}(t)|^2 + \frac{1}{p} \|u_{\varepsilon \mu m}(t)\|_{W^{1,p}_0(\Omega)}^p + \int_{0}^{t} \|u'_{\varepsilon \mu m}(s)\|^2_{H^1_0(\Omega)} ds \\
+ \frac{1}{\varepsilon} \int_{0}^{t} (\beta(u_{\varepsilon \mu m}'(s)), u_{\varepsilon \mu m}'(s)) ds + \frac{1}{\mu} \int_{0}^{t} (M(t) u_{\varepsilon \mu m}'(s), u_{\varepsilon \mu m}'(s)) ds \leq C
\end{align*}
$$

where $C$ is a positive constant independent of $\varepsilon, \mu, m$ and $t \in [0, t_m)$. Hence we can extend the solution $u_{\varepsilon \mu m}(t)$ to the whole interval $[0, T]$, obtaining in addition

\begin{align*}
(u_{\varepsilon \mu m}) &\text{ is bounded in } L^{\infty}(0, T; W^{1,p}_0(\Omega)), \\
(u_{\varepsilon \mu m}') &\text{ is bounded in } L^{\infty}(0, T; L^2(\Omega)), \\
(u_{\varepsilon \mu m}') &\text{ is bounded in } L^2(0, T; H^1_0(\Omega)), \\
(u_{\varepsilon \mu m}(T)) &\text{ is bounded in } W^{1,p}_0(\Omega), \\
(u_{\varepsilon \mu m}(T)') &\text{ is bounded in } L^2(\Omega).
\end{align*}

(5.5) - (5.9)
Since from (5.1) that $P$ the self-adjoint extension of $P$ through a standard argument on projections. Consider the projection operator where (5.3. A priori estimate II. We can thus extract subsequences from above sequences, denoted up to subindexes, such that

$$u_{\varepsilon \mu} \rightharpoonup^* u_{\varepsilon \mu} \quad \text{in} \quad L^\infty(0,T;W_0^{1,p}(\Omega)), \quad (5.13)$$

$$u_{\varepsilon \mu} \rightharpoonup u_{\varepsilon \mu} \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)), \quad (5.14)$$

$$u_{\varepsilon \mu} \rightharpoonup u_{\varepsilon \mu} \quad \text{in} \quad L^2(0,T;H_0^1(\Omega)), \quad (5.15)$$

$$u_{\varepsilon \mu}(T) \rightharpoonup^* u_{\varepsilon \mu}(T) \quad \text{in} \quad W_0^{1,p}(\Omega), \quad (5.16)$$

$$u_{\varepsilon \mu}(T) \rightharpoonup u_{\varepsilon \mu}(T) \quad \text{in} \quad L^2(\Omega). \quad (5.17)$$

Since $M \in L^\infty(\Omega_0)$, it follows from (5.14) that

$$\frac{1}{\sqrt{\mu}}Mu_{\varepsilon \mu} \rightharpoonup^* \frac{1}{\sqrt{\mu}}Mu_{\varepsilon \mu} \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)), \quad (5.18)$$

$$\beta(u_{\varepsilon \mu}) \rightharpoonup_{\varepsilon \mu} \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)), \quad (5.19)$$

$$\Delta_p u_{\varepsilon \mu} \rightharpoonup^* \varphi_{\varepsilon \mu} \quad \text{in} \quad L^\infty(0,T;W^{-1,p'}(\Omega)). \quad (5.20)$$

### 5.3. A priori estimate II.

Now we obtain an estimate for $u_{\varepsilon \mu}''$. It is done through a standard argument on projections. Consider the projection operator $P_m : H_0^\mu(\Omega) \to V_m$ defined by

$$P_m[h] = \sum_{j=1}^{m} ((h,w_j))w_j, \quad h \in H_0^\mu(\Omega)$$

where $(\cdot,\cdot)$ stands for the inner product in $H_0^\mu(\Omega)$. Let $P_m^* \in \mathcal{L}(H^{-\tau}(\Omega),H^{-r}(\Omega))$ the self-adjoint extension of $P_m$. Since $P_m^*[h] = P_m[h] = h$, $\forall h \in V_m$, we conclude from (5.1) that

$$\langle u_{\varepsilon \mu}'', t \rangle = \langle P_m^*[\Delta_p u_{\varepsilon \mu}''], w \rangle + \langle P_m^*[\Delta u_{\varepsilon \mu}''], w \rangle + \frac{1}{\varepsilon} \langle \beta(u_{\varepsilon \mu}''), w \rangle - \frac{1}{\mu} \langle P_m[M(t)u_{\varepsilon \mu}''], w \rangle \quad \forall w \in V_m.$$ 

Thereby, using argument of denseness it follows from (5.7), (5.10), (5.11) and (5.12) that

$$\langle u_{\varepsilon \mu}'' \rangle \text{ is bounded in } L^2(0,T;H^{-r}(\Omega)) \text{ for each } \varepsilon, \mu. \quad (5.21)$$

Taking into account the convergence obtained above, we can pass to the limit when $m \to \infty$ in the approximated equation and obtain

$$u_{\varepsilon \mu}'' + \varphi_{\varepsilon \mu} - \Delta u_{\varepsilon \mu}' + \frac{1}{\varepsilon} \chi_{\varepsilon \mu} + \frac{1}{\mu} Mu_{\varepsilon \mu} = \tilde{f} \quad \text{in} \quad L^2(0,T;W^{-1,p'}(\Omega)), \quad (5.22)$$

$$u_{\varepsilon \mu}(0) = \tilde{u}_0,$n

$$u_{\varepsilon \mu}'(0) = \tilde{u}_1.$$
It can be shown through the same arguments as in Ferreira-Ma [7] that \( \varphi_{\varepsilon \mu} = \Delta_p u_{\varepsilon \mu} \) and reasoning likewise as in Rabello [17] that \( \chi_{\varepsilon \mu} = \beta(u'_{\varepsilon \mu}) \). Therefore, we obtain

\[
\begin{align*}
\frac{1}{\varepsilon} u''_{\varepsilon \mu} + \Delta_p u_{\varepsilon \mu} - \Delta u'_{\varepsilon \mu} + \frac{1}{\varepsilon} \beta(u'_{\varepsilon \mu}) + \frac{1}{\mu} M u'_{\varepsilon \mu} = V \tilde{f} & \quad \text{in } L^2(0, T; W^{-1, p'}(\Omega)), \\
\varepsilon u_{\varepsilon \mu}(0) &= \tilde{u}_0, \\
\varepsilon u'_{\varepsilon \mu}(0) &= \tilde{u}_1. 
\end{align*}
\]

(5.22)

We observe that the bounds obtained are independently on \( \varepsilon, \mu \) and \( t \), thus there exist subsequences from previous sequences such that

\[
\begin{align*}
\varepsilon u_{\varepsilon \mu} & \xrightarrow{\varepsilon \to 0} u_{\varepsilon \mu} \quad \text{in } L^\infty(0, T; W^{1, p}_0(\Omega)), \\
\varepsilon u'_{\varepsilon \mu} & \xrightarrow{\varepsilon \to 0} u'_{\varepsilon \mu} \quad \text{in } L^\infty(0, T; L^2(\Omega)), \\
\varepsilon u''_{\varepsilon \mu} & \xrightarrow{\varepsilon \to 0} u''_{\varepsilon \mu} \quad \text{in } L^2(0, T; H^1_0(\Omega)), \\
\frac{1}{\sqrt{\mu}} M u'_{\varepsilon \mu} & \xrightarrow{\varepsilon \to 0} \frac{1}{\sqrt{\mu}} M u'_{\mu} \quad \text{in } L^\infty(0, T; L^2(\Omega)), \\
\int_0^T (\beta(u'_{\varepsilon \mu}), u'_{\varepsilon \mu}) dt & \xrightarrow{\varepsilon \to 0} 0. 
\end{align*}
\]

(5.23)-(5.26)

From (5.22) we obtain

\[
\beta(u'_{\varepsilon \mu}) = \varepsilon \left[ f - \Delta_p u_{\varepsilon \mu} - u''_{\varepsilon \mu} + \Delta u'_{\varepsilon \mu} - \frac{1}{\mu} M u'_{\varepsilon \mu} \right] \quad \text{in } \mathcal{D}'(0, T; W^{-1, p'}). 
\]

Thus, from the convergences (5.20), (5.23)-(5.26), it follows that

\[
\beta(u'_{\varepsilon \mu}) \xrightarrow{\varepsilon \to 0} 0 \quad \text{in } \mathcal{D}'(0, T; H^{-1}(\Omega)). 
\]

(5.27)

In addition, from (5.26), since \( \beta \) is Lipschitz,

\[
\beta(u'_{\varepsilon \mu}) \xrightarrow{\varepsilon \to 0} \chi \quad \text{in } L^2(0, T; H^{-1}(\Omega)). 
\]

Thereby we have \( \chi = 0 \). On the other hand, thanks to the monotonicity and hemicontinuity of \( \beta \) and (5.27), we prove that \( \chi = \beta(u_{\mu}) \) and therefore we conclude that

\[
\beta(u'_{\mu}(t)) = 0 \text{ a.e. or } u'_{\mu} \in K \text{ a.e.} 
\]

(5.28)
Let \( v \in L^2(0, T; W^{1,p}(\Omega)) \) such that \( v' \in L^2(0, T; W^{-1,p'}(\Omega)) \). Therefore,

\[
\frac{1}{2} |v(s) - u'_{\varepsilon\mu}(s)|^2 - \frac{1}{2} |v(0) - u'_{\varepsilon\mu}(0)|^2
\]

\[
= \frac{1}{2} \int_0^s \frac{d}{dt} |v(t) - u'_{\varepsilon\mu}(t)|^2 dt
\]

\[
= \int_0^s \langle v'(t) - u''_{\varepsilon\mu}(t), v(t) - u'_{\varepsilon\mu}(t) \rangle dt
\]

\[
= \int_0^s \langle v'(t) - \tilde{f}(t) - \Delta_p u_{\varepsilon\mu}(t) - \Delta u'_{\varepsilon\mu}(t) - \frac{1}{\varepsilon} \beta(u_{\varepsilon\mu}(t - \frac{1}{\varepsilon} \beta(u_{\varepsilon\mu}(t)) - \frac{1}{\mu} M u_{\varepsilon\mu}(t), v(t) - u'_{\varepsilon\mu}(t)) dt
\]

\[
= \int_0^s \langle v'(t) - \tilde{f}(t) - \Delta_p u_{\varepsilon\mu}(t) - \Delta u'_{\varepsilon\mu}(t), v(t) \rangle dt
\]

(5.29)

\[
+ \int_0^s (v'(t) - \tilde{f}(t), -u'_{\varepsilon\mu}(t)) dt
\]

\[
+ \int_0^s (\Delta_p u_{\varepsilon\mu}(t), u'_{\varepsilon\mu}(t)) dt - \int_0^s (\Delta u'_{\varepsilon\mu}(t), u'_{\varepsilon\mu}(t)) dt
\]

\[
+ \int_0^s \frac{1}{\varepsilon} \int_0^t (\beta(u'_{\varepsilon\mu}(t)), v(t) - u'_{\varepsilon\mu}(t)) dt + \int_0^s \frac{1}{\mu} (M(t) u'_{\varepsilon\mu}(t), v(t)) dt
\]

\[
+ \int_0^s \frac{1}{\mu} (M(t) u'_{\varepsilon\mu}(t), -u'_{\varepsilon\mu}(t)) dt.
\]

Let \( \Psi = \{ \varphi \in C^0[0,T], \varphi(t) \geq 0 \ \forall t \in [0,T] \} \). Multiplying (5.29) by \( \varphi \in \Psi \) and integrating from 0 to \( T \), we obtain

\[
\int_0^T \left[ \frac{1}{2} |v(s) - u'_{\varepsilon\mu}(s)|^2 + \frac{1}{p} ||u_{\varepsilon\mu}(s)||^2_{W^{1,p}} + \int_0^s ||u'_{\varepsilon\mu}(t)||^2 dt \right] \varphi(s) ds
\]

\[
\leq \int_0^T \varphi(s) \int_0^s \langle v'(t) - \tilde{f}(t) + \Delta_p u_{\varepsilon\mu}(t) - \Delta u'_{\varepsilon\mu}(t), v(t) \rangle dt ds
\]

\[
+ \int_0^T \varphi(s) \int_0^s \langle v'(t) - \tilde{f}(t), -u'_{\varepsilon\mu}(t) \rangle dt ds
\]

(5.30)

\[
+ \frac{1}{p} ||u_{\varepsilon\mu}(0)||^2_{W^{1,p}(\Omega)} \int_0^T \varphi(s) ds + \int_0^T \varphi(s) \int_0^s \frac{1}{\mu} (M(t) u'_{\varepsilon\mu}(t), v(t)) dt ds
\]

\[
+ \frac{1}{2} \int_0^T |v(0) - u'_{\varepsilon\mu}(0)|^2 \varphi(s) ds.
\]
Taking the limit inferior, it follows from (5.23)–(5.26) and from Banach-Steinhauss’ Theorem that
\[
\int_0^T \varphi(s) \left[ \frac{1}{2} |v(s) - u'_\mu(s)|^2 + \frac{1}{p} \|u_\mu(s)\|^2 + \int_0^s \|u'_\mu(t)\|^2 dt \right] ds
\leq \int_0^T \varphi(s) \left[ \int_0^s \langle v'(t) - \tilde{f}(t) + \Delta_p u'_\mu(t) - \Delta u'_\mu(t), v(t) \rangle dt ds + \int_0^T \varphi(s) \int_0^s \frac{1}{\mu} \langle M(t)u'_\mu(t), v(t) \rangle dt ds \right.
\]
\[
+ \int_0^T \varphi(s) \frac{1}{2} |v(0) - u_1|^2 ds, \quad \forall \varphi \in \Psi.
\]
Thus considering
\[
\varphi = \begin{cases} 
1 & \text{if } t = s, \\
\text{linear in } (s - \delta, s) \text{ and } (s, s + \delta),
\end{cases}
0 \leq s \leq 1, \varphi \in C^0[0,T],
\]
splitting the inequality (5.31) by \(\delta > 0\), taking the limit with \(\delta \to 0\), we obtain from the Lebesgue points Theorem for integrable functions
\[
\int_0^s \left[ \langle v'(t) - \tilde{f}(t) + \Delta_p u'_\mu(t) - \Delta u'_\mu(t), v(t) - u'_\mu(t) \rangle \right]
\]
\[
+ \frac{1}{\mu} \langle M(t)u'_\mu(t), v(t) \rangle dt
\geq \frac{1}{2} |v(s) - u'_\mu(s)|^2 - \frac{1}{2} |v(0) - u_1|^2, \quad \forall \mu, \text{ a.e.}
\]
We obtain, therefore, the penalized inequality in cylinder domain \(Q_0\), what proves Theorem 3.2.

5.4. **Passage to the limit.** It remains now passing to the limit when \(\mu \to 0\) to obtain the inequality in the noncylindrical domain \(Q\) and thus to have Theorem 3.1 proved.

From (5.23)–(5.26), Banach-Stainhaus’ Theorem and boundedness provided by (5.5), (5.6), (5.7) and (5.10) independently on \(\varepsilon\) and \(\mu\), there exist subsequences such that
\[
u_\mu \rightharpoonup u \quad \text{in } L^\infty(0,T;W^{1,p}_0(\Omega)),
\]
\[
u'_\mu \rightharpoonup u' \quad \text{in } L^\infty(0,T;L^2(\Omega)),
\]
\[
u'_\mu \rightharpoonup u' \quad \text{in } L^2(0,T;H^1_0(\Omega)),
\]
\[
1 \sqrt{\mu} M u'_\mu \rightharpoonup \chi_1 \quad \text{in } L^2(0,T;L^2(\Omega)).
\]
From (5.35) we obtain
\[
M u'_\mu \rightharpoonup \chi_2 \quad \text{in } L^2(0,T;H^1_0(\Omega)).
\]
We also have the convergence
\[
\beta u'_\mu \rightharpoonup \chi_3 \quad \text{in } L^\infty(0,T;L^2(\Omega)).
\]
Hence: for almost every $t$

$$M u'_\mu \rightharpoonup M u' \quad \text{in} \quad L^2(0, T; H^1_0(\Omega)).$$

Since $\frac{1}{\mu} \int_0^T |M(t)u'_\mu(t)|^2 dt \leq C \forall \mu$, it follows that $M u'_\mu \rightharpoonup 0$ in $L^2(Q_0)$. Hence,

$$M u' = 0 \quad \text{a.e. in } Q_0.$$

From the definition of $M$ we obtain

$$u' = 0 \quad \text{a.e. in } Q_0 \setminus Q$$

$$u' = 0 \quad \text{a.e. in } \Omega \setminus \Omega_t$$

in $[0, T]$, which, combined with (5.35), yields $u' \in L^2(0, T; H^1_0(\Omega_t))$. Since $u' = 0$ in $Q_0 \setminus Q$ and $u(x, 0) = u_0 = 0$ in $\Omega \setminus \Omega_0$, it follows that $u = 0$ in $Q_0 \setminus Q$, which jointly with (5.33),

$$u \in L^\infty(0, T; W_0^{1,p}(\Omega_t)). \quad (5.39)$$

Again, from de monotonicity and hemicontinuity of $\beta$, and owing to the fact that $\beta(u_\mu) = 0$ in $L^2(0, T; H^{-1}(\Omega))$, we conclude that

$$\beta(u') = 0 \quad \text{a.e. or } u'(t) \in K \text{ a.e. } (5.40)$$

We have $v \in L^2(0, T; W_0^{1,p}(\Omega_t)) \hookrightarrow L^2(0, T; H^1_0(\Omega_t))$. Let $v' \in L^2(0, T; W^{-1, p'}(\Omega_t))$. Hence: for almost every $t \in (0, T)$, $v = 0$ in $\Omega \setminus \Omega_t$. Thus

$$\int_0^s (M(t)u'_\mu(t), v)dt = \int_0^s \int_\Omega M(t)u'_\mu(t)v(t)dxdt$$

$$= \int_0^s \int_{\Omega_t} M(t)u'_\mu(t)v(t)dxdt = 0, \quad \forall \mu$$

because $M = 0$ in $\Omega_t$.

Taking the limit inferior in (5.32) in first member of the equation and the limit in the second member when $\mu \rightarrow 0$ and using the convergence obtained up to here, it follows that

$$\int_0^T \varphi(s) \left[ \frac{1}{2} |v(s) - u'(s)|^2 + \frac{1}{p} \|u(s)\|_{W_0^{1,p}(\Omega)}^p + \int_0^s \|u'(t)\|^2 dt \right]$$

$$\leq \int_0^T \varphi(s) \int_0^s \langle v' - f + u_\mu - \Delta u', v \rangle dt ds + \int_0^T \varphi(s) \int_0^s \langle v' - f, u' \rangle dt ds$$

$$+ \int_0^T \varphi(s) \int_0^s \|u(0)\|^2 ds + \int_0^T \varphi(s) \frac{1}{2} |v(0) - u_1|^2 ds.$$

Thus, for almost $s$ we have

$$\frac{1}{2} |v(s) - u'(s)|_L^2(\Omega_s) - \frac{1}{2} |v(0) - u'(0)|_L^2(\Omega_0)$$

$$\leq \int_0^s \langle v' - f + \Delta u - \Delta u', v \rangle dt + \int_0^s \langle v' - f, u' \rangle ds$$

$$+ \frac{1}{p} \|u(0)\|_{W_0^{1,p}(\Omega_0)}^p - \frac{1}{p} \|u(s)\|_{W_0^{1,p}(\Omega_s)}^p ds - \int_0^s \|u'(t)\|^2 dt$$

$$= \int_0^s \langle v' - f + \Delta u - \Delta u', v \rangle dt + \int_0^s \langle v' - f, u' \rangle ds$$

$$- \int_0^s \langle \Delta u, u' \rangle dt - \int_0^s \langle -\Delta u', u' \rangle dt.$$
\[
\begin{align*}
&= \int_0^s \langle v' - f + \Delta_p u - \Delta u', v \rangle \, dt + \int_0^s \langle v' - f + \Delta_p u - \Delta u', -u' \rangle \, dt \\
&= \int_0^s \langle v' - f + \Delta_p - \Delta u', v - u' \rangle \, dt
\end{align*}
\]
for all \( s \in (0, T) \), and all \( v \in V \).

To show the continuity of \( u' \) we can use the same arguments as in [11].

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